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The Equivalence of Stack-Counter Acceptors and Quasi-Realtime Stack-Counter Acceptors*

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It is shown that each language accepted by a stack-counter acceptor is accepted by a stack-counter acceptor operating in quasi-realtime.

INTRODUCTION

Two well-known devices studied in computer science are the pushdown acceptor [5] and the (one-way) stack acceptor [7]. Pushdown acceptors in which the pushdown alphabet consists of just one symbol are called counter acceptors and have been extensively studied in their own right [2–4, 10] as well as in AFL theory [8, 9]. It therefore seems reasonable to consider stack accaptors in which the auxiliary storage alphabet consists of just one symbol. In the present paper we do this, calling the ensuing devices stack-counter acceptors. Here our interest is not in the usual questions treated in language theory. Rather, we examine the devices in comparison with the quasi-realtime stack-counter acceptors, i.e., the stack-counter acceptors in which each accepted word can be accepted without more than a given number of consecutive ϵ-input moves.[1] (This type of problem came into prominence recently with the advent of AFL and AFA theory.) Our major result is the rather surprising statement indicated in the title to the paper, namely, that the languages accepted by arbitrary stack-counter

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[1] The languages accepted by these devices coincide with the languages accepted by stack-counter acceptors in which there are at most a given number of consecutive ϵ-input moves.

acceptors and the languages accepted by quasi-realtime stack-counter acceptors coincide. (More surprising, at least to the authors, is the length and complexity of the argument needed to demonstrate that fact.) This theorem has already been used elsewhere [1]. A modification of the proof elicits the fact that the languages accepted by arbitrary counter acceptors and the languages accepted by quasi-realtime counter acceptors coincide.

There are five sections to the paper. Section 1 introduces the stack-counter acceptors. Section 2 associates a particular finite-index congruence relation, used constantly in the remaining sections, with each stack-counter acceptor. Sections 3, 4, and 5 are devoted to the argument for the main result. Let $\mathscr{L}$ and $\mathscr{L}_{\mathrm{QR}}$ denote the families of languages accepted by stack-counter acceptors and quasi-realtime stack-counter languages, respectively. Obviously $\mathscr{L}_{\mathrm{QR}} \subseteq \mathscr{L}$. In Section 3 we introduce

(i) $\mathscr{L}_{\mathrm{QRG}}$, the family of languages accepted by "quasi-realtime generalized stack-counter acceptors,"

(ii) $\mathscr{L}_{\mathrm{BQRG}}$, the family of languages accepted by "bounded quasi-realtime generalized stack-counter acceptors," and

(iii) $\mathscr{L}_{\mathrm{SEQR}}$, the family of languages accepted by "SE-quasi-realtime stack-counter acceptors."

We then show (Proposition 3.1) that $\mathscr{L}_{\mathrm{QRG}} \subseteq \mathscr{L}_{\mathrm{SEQR}}$ and $\mathscr{L}_{\mathrm{BQRG}} \subseteq \mathscr{L}_{\mathrm{QR}}$. In Section 4, we show (Proposition 4.1) that $\mathscr{L} \subseteq \mathscr{L}_{\mathrm{QRG}}$. In Section 5, we show (Proposition 5.1) that $\mathscr{L}_{\mathrm{SEQR}} \subseteq \mathscr{L}_{\mathrm{BQRG}}$. Combining, we obtain $\mathscr{L} \subseteq \mathscr{L}_{\mathrm{QR}}$, thus $\mathscr{L} = \mathscr{L}_{\mathrm{QR}}$.

1. Preliminaries

In this section we present the one-way stack-counter acceptor (abbreviated "stack-counter acceptors") and introduce a formalism for working with them. In general, we assume that the reader has an elementary knowledge of formal language theory, as for example, in [5].

Intuitively, a stack-counter acceptor is a device with a read-only input tape, a finite control, and a stack storage[2] over a one-letter alphabet. Formally, we proceed as follows:

NOTATION. Let K and Σ be infinite sets (of *states* and *input symbols*, respectively). Let c be a new symbol (called the *storage* symbol). Let $\uparrow$ be a new symbol (called the *pointer*).

The sets K and Σ are to contain all possible states and all possible input symbols. The symbol c is to be the one letter over which the auxiliary storage is to be defined.

[2] A "stack" storage is an auxiliary storage which increases and decreases as a pushdown tape and whose interior can be read but not erased.

DEFINITION. A *configuration* is a triple $(p, w, \alpha\uparrow\beta)$, where p is in K, w is in[3] Σ^*, and $\alpha\beta$ is in c^*. p, w, $\alpha\uparrow\beta$, $\alpha\beta$, α, and β are called the *state*, *input*, *stack*, *stack word*, *left stack word*, and *right stack word*, respectively (of the configuration).

Intuitively, in configuration $(p, w, \alpha\uparrow\beta)$, the input head is scanning the first (if any) symbol of w. The storage pointer is a wedge between the last (if any) symbol of α and the first (if any) symbol of β. In addition, it has the ability to determine whether or not α is empty and whether or not β is empty.

We next describe different types of "moves," that is, instructions for transforming configurations.

DEFINITION. For each p in K, q in K, and a in Σ, let $I(p, q, a)$, $S(p, q)$, $E(p, q)$, $R(p, q)$, $L(p, q)$, $\$(p, q)$, $p \to q$, and $c(p, q)$ denote the following relations between configurations:

$$\begin{aligned}
I(p, q, a) &= \{((p, aw, \alpha\uparrow\beta), (q, w, \alpha\uparrow\beta))/w \text{ in } \Sigma^*, \alpha\beta \text{ in } c^*\},\\
S(p, q) &= \{((p, w, \alpha\uparrow), (q, w, \alpha c\uparrow))/w \text{ in } \Sigma^*, \alpha \text{ in } c^*\},\\
E(p, q) &= \{((p, w, \alpha c\uparrow), (q, w, \alpha\uparrow))/w \text{ in } \Sigma^*, \alpha \text{ in } c^*\},\\
R(p, q) &= \{((p, w, \alpha\uparrow c\beta), (q, w, \alpha c\uparrow\beta))/w \text{ in } \Sigma^*, \alpha\beta \text{ in } c^*\},\\
L(p, q) &= \{((p, w, \alpha c\uparrow\beta), (q, w, \alpha\uparrow c\beta))/w \text{ in } \Sigma^*, \alpha\beta \text{ in } c^*\},\\
\$(p, q) &= \{((p, w, \alpha\uparrow), (q, w, \alpha\uparrow))/w \text{ in } \Sigma^*, \alpha \text{ in } c^*\},\\
p \to q &= \{((p, w, \alpha\uparrow\beta), (q, w, \alpha\uparrow\beta))/w \text{ in } \Sigma^*, \alpha\beta \text{ in } c^*\},\\
¢(p, q) &= \{((p, w, \uparrow\beta), (q, w, \uparrow\beta))/w \text{ in } \Sigma^*, \beta \text{ in } c^*\}.
\end{aligned}$$

Each of the above relations is called a *move*. $I(p, q, a)$ is called an *input* move, $S(p, q)$ a *store* move, $E(p, q)$ an *erase* move, $R(p, q)$ a *right shift* (*move*), $L(p, q)$ a *left shift* (*move*), $\$(p, q)$ a *right end move*, $p \to q$ a *state change* (*move*), and $¢(p, q)$ a *left end move.* Any move other than an input move is called an ϵ-*move*.

Intuitively, $I(p, q, a)$ is the instruction: "If the first input symbol is a, then erase it and go from state p to q." $S(p, q)$ $(E(p, q))$ is the instruction: "If the storage pointer is at the right end of the stack word (and the stack word is non-ϵ) ,then go from p to q and store one c (erase one c) in the left stack word." $R(p, q)$ $(L(p, q))$ is the instruction: "If the right (left) stack word is non-ϵ, then go from p to q and move the storage pointer one symbol to the right (left)." $\$(p, q)$ $(¢,(p, q))$ is the instruction: "If the storage pointer is at the right (left) end, then go from p to q." $p \to q$ is the instruction: "Go from p to q." Note that, with respect to applicability as well as effect produced,

[3] For each set E, E^* is the free semigroup with identity ϵ. Each element of E is called a *word* (of E^*).

input moves are independent of the stack words and ϵ-moves are independent of the input.

NOTATION. Let ξ and η be configurations. For each move μ, write $\xi \vdash_\mu \eta$ if (ξ, η) is in μ. For $n \geqslant 1$ and moves $\mu_1, \ldots, \mu_n$, write $\xi_1 \vdash_{\mu_1} \cdots \vdash_{\mu_n} \xi_{n+1}$ if $\xi_i \vdash_{\mu_i} \xi_{i+1}$ for each i. Let M be a set of moves. Write $\xi \vdash_M \eta$ if $\xi \vdash_\mu \eta$ for some μ in M. Let $\xi \vdash_M^0 \xi$ and, for $n \geqslant 1$, write $\xi \vdash_M^n \eta$ if $\xi = \xi_1 \vdash_M \cdots \vdash_M \xi_{n+1} = \eta$. Write $\xi \vdash_M^* \eta$ if $\xi \vdash_M^n \eta$ for some $n \geqslant 0$.

We are now ready to define a stack-counter acceptor.

DEFINITION. A *stack-counter acceptor* is a triple $S = (M, q_0, q_1)$, where M is a finite set of moves and q_0 (called the *start* state) and q_1 (called the *accepting* state) are in K.

Observe that each stack-counter acceptor uses only finitely many states and input symbols.

A stack-counter acceptor specifies a set of words as follows:

DEFINITION. Let $S = (M, q_0, q_1)$ be a stack-counter acceptor. A word w in Σ^* is said to be *accepted* by S if $(q_0, w, \uparrow) \vdash_M^* (q_1, \epsilon, \uparrow)$. Let $L(S)$ denote the set of all words accepted by S. A subset $L \subseteq \Sigma^*$ is said to be a *stack-counter language* if $L = L(S)$ for some stack-counter acceptor.[4] Let $\mathscr{L}$ denote the family of all stack-counter languages.

Let $S = (M, q_0, q_1)$ and $\Sigma_1 = \{a/I(p, q, a)$ in M for some p and $q\}$. Then Σ_1 is finite and $L(S) \subseteq \Sigma_1{}^*$. Thus each stack-counter language is a set of words over a finite alphabet.

Stack-counter acceptors can be used in another manner to define a family of languages.

DEFINITION. For $d \geqslant 0$, $S = (M, q_0, q_1)$ is said to be *quasi-realtime with delay d* if, for each word w in $L(S)$, there exists some sequence $\mu_1, \ldots, \mu_n$ of moves in M such that $(q_0, w, \uparrow) \vdash_{\mu_1} \cdots \vdash_{\mu_n} (q_1, \epsilon, \uparrow)$ and there are no more than d consecutive ϵ-moves, i.e., if $\mu_{k+1}, \ldots, \mu_{k+j}$ are ϵ-moves, then $j \leqslant d$. S is said to be *quasi-realtime* if S is quasi-realtime with delay d for some $d \geqslant 0$. Let $\mathscr{L}_{QR}$ be the family of all quasi-realtime stack-counter acceptor languages.

As mentioned in the introduction, the purpose of the present paper is to prove that

(*) $\mathscr{L} = \mathscr{L}_{QR}$, i.e., each stack-counter language is a quasi-realtime stack-counter language, and conversely.

[4] In general, if S is an "X-acceptor" and $L(S)$ is defined, then $L(S)$ is called an "X-acceptor language."

In particular, we shall need three main auxiliary results to prove (*), namely, Propositions 3.1, 4.1, and 5.1. In addition, we shall need a special congruence relation mentioned in Section 2.

We conclude this section by discussing variations in some of the previous concepts.

Remarks. (1) Call $S = (M, q_0, F)$ a "modified stack-counter acceptor' 'if M is a finite set of moves, q_0 is in K, and $F \subseteq K$. Let $L(S) = \{w/(q_0, w, \uparrow) \vdash_M^* (q, \epsilon, \uparrow)$ for some q in $F\}$. The notion of "quasi-realtime" modified stack-counter acceptor is to be the obvious one. Clearly, each L in $\mathscr{L}(\mathscr{L}_{\mathrm{QR}})$ is a (quasi-realtime) modified stack-counter acceptor language. The converse is also true. In particular, let $S = (M, q_0, F)$ be a modified stack-counter acceptor. Let $S' = (M', q_0, q_1)$, where q_1 is a new symbol and $M' = M \cup \{\$(q, q_1)/q$ in $F\}$. Then $L(S') = L(S)$, and S' is quasi-realtime if S is.

(2) Given a stack-counter acceptor $S = (M, q_0, q_1)$, let

$$L_f(S) = \{w/(q_0, w\uparrow) \vdash_M^* (q_1, \epsilon, \alpha\uparrow\beta) \text{ for some } \alpha \text{ and } \beta\}.$$

Thus $L_f(S)$ is the set of words defined by "acceptance by final state". The family of languages so obtained is $\mathscr{L}$. [For suppose $S = (M, q_0, q_1)$. Let $S' = (M', q_0, q_2)$, where $M' = M \cup \{R(q_1, q_2), R(q_2, q_2), E(q_2, q_2)\}$. Then $L(S') = L_f(S)$. Let $S'' = (M'', q_0, q_3)$, where q_2 and q_3 are new symbols in K and $M'' = M \cup \{¢(q_1, q_2), \$(q_2, q_3)\}$. Then $L_f(S'') = L(S)$.] A similar statement holds for the quasi-realtime case. [For again consider $S = (M, q_0, q_1)$. The construction above for S'' shows that S'' is quasi-realtime if S is quasi-realtime, and $L_f(S'') = L(S)$. Also, there exists S' (not necessarily quasi-realtime) such that $L(S') = L_f(S)$. By (*), there exists a quasi-realtime stack-counter acceptor S''' such that $L(S''') = L_f(S)$.]

(3) Given $d \geqslant 0$, call $S = (M, q_0, q_1)$ "strongly quasi-realtime with delay d" if

$$(p, \epsilon, \alpha \uparrow \beta) \vdash_M^m (p', \epsilon, \alpha' \uparrow \beta')$$

implies $m \leqslant d$, for all p, p', α, β, α', and β'. Call S "strongly quasi-realtime" if S is strongly quasi-realtime with delay d for some $d \geqslant 0$.[5] Obviously, if S is strongly quasi-realtime, then $L(S)$ is in $\mathscr{L}_{\mathrm{QR}}$. The converse is also true. [For suppose that $S = (M, q_0, q_1)$ is quasi-realtime with delay d. For each state q of S,[6] let $(q, 0),\ldots, (q, d)$ be $d + 1$ new states. Let M' consist of the following moves: For each input move $I(p, q, a)$ in M let $I((p, i), (q, 0), a)$, $0 \leqslant i \leqslant d$, be in M'. For each $1 \leqslant i < d$ and each ϵ-move $U(p, q)$ in M where $U(p, q)$ is one of the forms $S(p, q)$, $E(p, q)$, $R(p, q)$, $L(p, q)$, $p \to q$, $¢(p, q)$, or $\$(p, q)$, let $U((p, i), (q, i + 1))$ be in M'. Let $(q_1, i) \to q_1$ be in M' for each i, $0 \leqslant i \leqslant d$. Then $S' = (M', (q_0, 0), q_1)$ is strongly quasi-realtime and $L(S) = L(S')$.]

[5] The notion of "strongly quasi-realtime" was first introduced in [6], under the term "quasi-realtime," and has received prominent attention ever since.

[6] Each state in a move of M is called a *state of S*.

(4) Consider one-way stack acceptors [7] with configurations of the form $(p, w, ¢\alpha w\rceil \beta\$)$. Suppose further that α and β are words in c^*. Then, except for inessential changes, the configuration $(p, w, ¢\alpha\rceil \beta\$)$ becomes the configuration $(p, q, \alpha\uparrow\beta)$ in the stack-counter acceptor. The rules $¢(p, q)$ and $\$(p, q)$ allow recognition of the left and right ends of the stack, as is done in the one-way stack acceptor, by

$$(p, w, ¢\rceil \alpha\beta\$) \vdash (q, w, ¢\rceil \alpha\beta\$) \qquad \text{and} \qquad (p, w, ¢\alpha\beta\$\rceil) \vdash (q, w, ¢\alpha\beta\$\rceil),$$

respectively. In other words, a stack-counter acceptor is essentially a one-way stack-acceptor with a storage alphabet of one symbol and the ability to recognize both the left and right ends of the stack. It is easy to show that removal of the moves $\$(p, q)$ does not decrease the resulting family of languages. It is an open question (although the authors strongly suspect that the answer is positive) whether removing the rules $¢(p, q)$ changes the family of languages.

2. A Fundamental Congruence

In this section we associate with each stack-counter acceptor S a finite-index congruence relation $\equiv_S$ on c^*. This congruence relation plays an essential role in the remaining sections.

We recall the following well-known concepts.

Definition. Let Σ_1 be a finite set. A *congruence relation* on Σ_1^* is an equivalence relation $\equiv$ which is both left and right invariant, i.e., if $x \equiv y$ then $uxv \equiv uyv$ for all u and v in Σ_1^*.

As usual, we shall let $[\alpha]$ denote the equivalence class containing α.

Definition. A congruence relation $\equiv$ is of *finite index* if there are only a finite number of equivalence classes generated by $\equiv$.

We also recall the following well-known result [11].

Proposition 2.1. *A set $U \subseteq c^*$ is regular if and only if there is a finite-index congruence relation $\equiv$ on c^* such that α in U and $\beta \equiv \alpha$ imply β in U.*

We now consider a canonical form for each equivalence class of a finite-index congruence relation on c^*.

Notation. Let $\leqslant$ and $<$ be the relations on c^* defined as follows. For α and β in c^* write $\alpha \leqslant \beta (\alpha < \beta)$ if[7] $|\alpha| \leqslant |\beta|$ $(|\alpha| < |\beta|)$. For α and β in c^*, let β/α (also

[7] For each word u, $|u|$ denotes its length.

written $\frac{\beta}{\alpha}$) denote the word $c^{|\beta|-|\alpha|}$ if $\alpha \leqslant \beta$ and let β/α be undefined otherwise. For $A \subseteq c^*$ and $B \subseteq c^*$, let B/A (also written $\frac{B}{A}$) denote the set of all β/α, where α is in A and β is in B.

The following result is a restatement of the well-known theorem for finite cyclic semigroups.

PROPOSITION 2.2. *For each finite-index congruence relation on c^*, there exist unique words π and κ such that every congruence class is either the unit set $\{\alpha\}$ for some $\alpha < \kappa$ or is $\alpha\pi^*$ for some α, $\kappa \leqslant \alpha < \pi\kappa$.*

The words π and κ in Proposition 2.2 will be called the *period* and *constant*, respectively, of the congruence relation.

Remark. Note that π cannot be ϵ. However, κ might be ϵ. For example, let the equivalence classes be $(c^2)^*$ and $c(c^2)^*$. Then $\pi = c^2$ and $\kappa = \epsilon$.

We now turn to a congruence relation $\equiv_S$ for each stack-counter acceptor $S = (M, q_0, q_1)$.

NOTATION. Let S be a stack-counter acceptor and σ a word in c^*. Write

$$\xi_0 \,\underset{\mu_1}{\overline{\vdash}} \cdots \underset{\mu_n}{\overline{\vdash}}\, \xi_n \textit{ over } \sigma$$

if $\xi_0 \vdash_{\mu_1} \cdots \vdash_{\mu_n} \xi_n$ and each configuration ξ_i has stack word $\geqslant \sigma$. Let M be a set of moves. Write

$$\xi \overset{n}{\underset{M}{\vdash}} \eta \textit{ over } \sigma$$

if $\xi_0 \vdash_{\mu_1} \cdots \vdash_{\mu_n} \eta$ over σ for some sequence $\mu_1, \ldots, \mu_n$ of moves in M. Write

$$\xi \overset{*}{\underset{M}{\vdash}} \eta \textit{ over } \sigma$$

if $\xi \vdash^n_M \eta$ over σ for some $n \geqslant 0$.

NOTATION. Let K_0 be the set of states of S and Σ_0 the set of input symbols of[8] S. Let M_0 be the set of all moves formed from K_0 and Σ_0 (and c) i.e., $M_0 = \{I(p, q, a), S(p, q), R(p, q), L(p, q), \$(p, q), p \to q, ¢(p, q) / p$ and q in K_0, a in $\Sigma_0\}$. Let $M_{RL\$¢\to} = \{R(p, q), L(p, q), \$(p, q), ¢(p, q), p \to q / p$ and q in $K_0\}$.

In order to define the congruence relation $\equiv_S$, mentioned in the beginning of the section, we first introduce a preliminary relation $\equiv_1$.

[8] The *set of input symbols of* $S = (M, q_0, q_1)$ is defined as the set $\{a / I(p, q, a)$ in M for some p and $q\}$.

NOTATION. Let $\equiv_1$ be the relation on c^* defined by $\beta \equiv \bar{\beta}$ if, for all $N \subseteq M_{RL\$¢\rightarrow}$, p in K_0, and q in K_0:

(α) $(p, \epsilon, \uparrow\beta) \vdash_N^* (q, \epsilon, \beta\uparrow)$ if and only if $(p, \epsilon, \uparrow\bar{\beta}) \vdash_N^* (q, \epsilon, \bar{\beta}\uparrow)$,

(β) $(p, \epsilon, \beta\uparrow) \vdash_N^* (q, \epsilon, \uparrow\beta)$ if and only if $(p, \epsilon, \bar{\beta}\uparrow) \vdash_N^* (q, \epsilon, \uparrow\bar{\beta})$,

(γ) $(p, \epsilon, \beta\uparrow) \vdash_N^* (q, \epsilon, \beta\uparrow)$ if and only if $(p, \epsilon, \bar{\beta}\uparrow) \vdash_N^* (q, \epsilon, \bar{\beta}\uparrow)$, and

(δ) $(p, \epsilon, \uparrow\beta) \vdash_N^* (q, \epsilon, \uparrow\beta)$ if and only if $(p, \epsilon, \uparrow\bar{\beta}) \vdash_N^* (q, \epsilon, \uparrow\bar{\beta})$.

Example. Let $K_0 = \{p, q\}$. Then $[c]_{\equiv_1} = \{c\}$. [Obviously $c \not\equiv_1 \epsilon$. Let $N = \{R(p, q)\}$. Then $\{p, \epsilon, \uparrow c) \vdash_N^* (q, \epsilon, c\uparrow)$. For $\bar{\beta} = c\beta'$, $\beta' \neq \epsilon$, $(p, \epsilon, \uparrow c\beta') \vdash_N (q, \epsilon, c\uparrow\beta')$ holds but $(q, \epsilon, c\uparrow\beta') \vdash_N^* (q, \epsilon, c\beta'\uparrow)$ is false.] On the other hand, it can be shown that $[c^2]_{\equiv_1} = c^2(c^2)^*$.

Two facts about $\equiv_1$ are the following:

PROPOSITION 2.3. (a) *$\equiv_1$ is a finite-index congruence relation, with non-ϵ constant.*

(b) *Let $\alpha \equiv \bar{\alpha}$. Then, for all $N \subseteq M_0$, p in K_0, q in K_0, β in c^*, and γ in c^*,*

$(p, \epsilon, \alpha\beta\uparrow) \vdash_{\overline{N}}^* (q, \epsilon, \alpha\gamma\uparrow)$ *over* α *if and only if* $(p, \epsilon, \bar{\alpha}\beta\uparrow) \vdash_{\overline{N}}^* (q, \epsilon, \bar{\alpha}\gamma\uparrow)$ *over* $\bar{\alpha}$.

Proof. We shall show only (a), the proof for (b) being straightforward.

Obviously $\equiv_1$ is a congruence relation. That the constant is non-ϵ follows from the the fact that, for any $\beta \neq \epsilon$ and p in K_0,

$$(p, \epsilon, \uparrow\epsilon) \vdash_{\overline{\phi}}^* (p, \epsilon, \epsilon\uparrow)$$

holds but

$$(p, \epsilon, \uparrow\beta) \vdash_{\overline{\phi}}^* (p, \epsilon, \beta\uparrow)$$

does not.

Now consider the number of equivalence classes of $\equiv_1$. Let m and k be the number of elements in $M_{RL\$¢\rightarrow}$ and K_0, respectively. For each word β in c^* there are $4k^2 2^m$ questions having one of the following forms (where N is an arbitrary subset of $M_{RL\$¢\rightarrow}$):

(a) Is $(p, \epsilon, \uparrow\beta) \vdash_N^* (q, \epsilon, \beta\uparrow)$?

(b) Is $(p, \epsilon, \beta\uparrow) \vdash_N^* (q, \epsilon, \uparrow\beta)$?

(c) Is $(p, \epsilon, \beta\uparrow) \vdash_N^* (q, \epsilon, \beta\uparrow)$?

(d) Is $(p, \epsilon, \uparrow\beta) \vdash_N^* (q, \epsilon, \uparrow\beta)$?

Clearly $\alpha \equiv_1 \alpha'$ if and only if the answers to each of these $4k^2 2^m$ questions is the same for $\beta = \alpha$ as for $\beta = \alpha'$. Since there are at most $2^{4k^2 2^m}$ sets of answers, there are at most $2^{4k^2 2^m}$ equivalence classes.

NOTATION. Let π_1 be the period and κ_1 the constant of $\equiv_1$. Let $\equiv_S$ be the relation on c^* defined by $\beta \equiv_S \bar{\beta}$ if $\beta \equiv_1 \bar{\beta}$ and for all $N \subseteq M_0$, p in K_0, and $\alpha < \pi_1\kappa_1$ in c^*:

(α) $(p, \epsilon, \alpha\uparrow) \vdash_N^* (q, \epsilon, \alpha\beta\uparrow)$ over α if and only if $(p, \epsilon, \alpha\uparrow) \vdash_N^* (q, \epsilon, \alpha\bar{\beta}\uparrow)$ over α;

and

(β) $(p, \epsilon, \alpha\beta\uparrow) \vdash_N^* (q, \epsilon, \alpha\uparrow)$ over α if and only if $(p, \epsilon, \alpha\bar{\beta}\uparrow) \vdash_N^* (q, \epsilon, \alpha\uparrow)$ over α.

PROPOSITION 2.4. *$\equiv_S$ is a finite index congruence relation on c^* with the following properties (for each p, q in K_0, $\alpha \equiv_S \bar{\alpha}$, $\beta \equiv_S \bar{\beta}$, and $\gamma \equiv_S \bar{\gamma}$):*

(a) $(p, \epsilon, \alpha\uparrow) \vdash_M^* (q, \epsilon, \alpha\beta\uparrow)$ *over* α *if and only if* $(p, \epsilon, \bar{\alpha}\uparrow) \vdash_M^* (q, \epsilon, \bar{\alpha}\bar{\beta}\uparrow)$ *over* $\bar{\alpha}$,

(b) $(p, \epsilon, \alpha\beta\uparrow) \vdash_M^* (q, \epsilon, \alpha\uparrow)$ *over* α *if and only if* $(p, \epsilon, \bar{\alpha}\bar{\beta}\uparrow) \vdash_M^* (q, \epsilon, \bar{\alpha}\uparrow)$ *over* $\bar{\alpha}$,

(c) $(p, \epsilon, \alpha\uparrow\gamma\beta) \vdash_M^* (q, \epsilon, \alpha\gamma\uparrow\beta)$ *over* $\alpha\gamma\beta$ *if and only if* $(p, \epsilon, \bar{\alpha}\uparrow\bar{\gamma}\bar{\beta}) \vdash_M^* (q, \epsilon, \bar{\alpha}\bar{\gamma}\uparrow\bar{\beta})$ *over* $\bar{\alpha}\bar{\gamma}\bar{\beta}$,

(d) $(p, \epsilon, \alpha\gamma\uparrow\beta) \vdash_M^* (q, \epsilon, \alpha\uparrow\gamma\beta)$ *over* $\alpha\gamma\beta$ *if and only if* $(p, \epsilon, \bar{\alpha}\bar{\gamma}\uparrow\bar{\beta}) \vdash_M^* (q, \epsilon, \bar{\alpha}\uparrow\bar{\gamma}\bar{\beta})$ *over* $\bar{\alpha}\bar{\gamma}\bar{\beta}$, *and*

(e) *the constant of* $\equiv_S$ *is non-*ϵ.

Proof. We shall only show that $\equiv_S$ if of finite index, the remaining items being straightforward.

Given two words α and α' in Σ^*, $\alpha \equiv_S \alpha'$ if and only if $\beta = \alpha$ and $\beta = \alpha'$ have the same answer for each of the $4k^2 2^m$ questions in the proof of Proposition 2.3 and each of the following questions (for arbitrary $N \subseteq M_0$, p in K_0, q in K_0, and $\alpha < \pi_1\kappa_1$):

(a) Is $(p, \epsilon, \alpha\uparrow) \vdash_N^* (q, \epsilon, \alpha\beta\uparrow)$ over α?

(b) Is $(p, \epsilon, \alpha\beta\uparrow) \vdash_N^* (q, \epsilon, \alpha\uparrow)$ over α?

There are only $2k^2 2^{m_0} \mid \pi_1\kappa_1 \mid$ questions of (a) or (b), where $m_0 = \#(M_0)$. Thus there are at most $2^{4k^2 2^m + 2k^2 2^{m_0} |\pi_1\kappa_1|}$ sets of answers, so that $\equiv_S$ is of finite index.

3. GENERALIZED STACK-COUNTER ACCEPTORS

In the present section we introduce three new families of languages. These are

(i) the family of quasi-realtime generalized stack-counter acceptor languages, abbreviated $\mathscr{L}_{\mathrm{QRG}}$,

(ii) the family of "bounded quasi-realtime generalized stack-counter acceptor languages," abbreviated $\mathscr{L}_{\mathrm{BQRG}}$, and

(iii) the family of "SE-quasi-realtime stack-counter acceptor languages," abbreviated $\mathscr{L}_{\mathrm{SEQR}}$.

We then prove the first of the three main auxiliary results, namely, that $\mathscr{L}_{\mathrm{QRG}} \subseteq \mathscr{L}_{\mathrm{SEQR}}$ and $\mathscr{L}_{\mathrm{BQRG}} \subseteq \mathscr{L}_{\mathrm{QR}}$.

To define the first of the above three families, we now extend the class of store moves, erase moves, left shifts, right shifts, and left end moves.

DEFINITION. For each p in K, q in K, a in Σ, regular subsets A, B, and C of c^*, and words α and γ in c^*, let $I(p, q, a)$ be as in Section 1 and

$$\begin{aligned}
S(p, q, A, C) &= \{((p, w, \alpha\uparrow), (q, w, \alpha\gamma\uparrow)/w \text{ in } \Sigma^*, \alpha \text{ in } A, \gamma \text{ in } C\},\\
E(p, q, A, C) &= \{((p, w, \alpha\gamma\uparrow), (q, w, \alpha\uparrow)/w \text{ in } \Sigma^*, \alpha \text{ in } A, \gamma \text{ in } C\},\\
R(p, q, A, \gamma, B) &= \{((p, w, \alpha\uparrow\gamma\beta), (q, w, \alpha\gamma\uparrow\beta))/w \text{ in } \Sigma^*, \alpha \text{ in } A, \beta \text{ in } B\},\\
L(p, q, A, \gamma, B) &= \{((p, w, \alpha\gamma\uparrow\beta), (q, w, \alpha\uparrow\gamma\beta))/w \text{ in } \Sigma^*, \alpha \text{ in } A, \beta \text{ in } B\},\\
¢S(p, q, \gamma, \alpha, B) &= \{((p, w, \alpha\uparrow\beta), (q, w, \gamma\alpha\uparrow\beta))/w \text{ in } \Sigma^*, \beta \text{ in } B\}, \text{ and}\\
¢E(p, q, \gamma, \alpha, B) &= \{((p, w, \gamma\alpha\uparrow\beta), (q, w, \alpha\uparrow\beta))/w \text{ in } \Sigma^*, \beta \text{ in } B\}.
\end{aligned}$$

Each of the above seven sets is called a *generalized move*. $I(p, q, a)$ is called a *generalized input move*, $S(p, q, A, C)$ a *generalized store move*, $E(p, q, A, C)$ a *generalized erase move*, $R(p, q, A, \gamma, B)$ *a generalized right shift move*, $L(p, q, A, \gamma, B)$ a *generalized left shift move*, $¢S(p, q, \gamma, \alpha, B)$ a *left end store move*, and $¢E(p, q, \gamma, \alpha, B)$ a *left end erase move*. Moves other than generalized input moves are called *generalized ϵ-moves*.

Note that every move is a generalized move. In particular, $S(p, q) = S(p, q, c^*, \{c\})$, $E(p, q) = E(p, q, c^*, \{c\})$, $R(p, q) = R(p, q, c^*, c, c^*)$, $L(p, q) = L(p, q, c^*, c, c^*)$, $\$(p, q) = S(p, q, c^*, \{\epsilon\})$, $p \to q = R(p, q, c^*, \epsilon, c^*)$, and $¢(p, q) = ¢S(p, q, \epsilon, \epsilon, c^*)$.

DEFINITION. A *generalized stack-counter acceptor* is a triple $S = (M, q_0, q_1)$ where M is a finite set of generalized moves and q_0 and q_1 are in K. A generalized stack-counter acceptor is called *bounded* if, for every store move $S(p, q, A, C)$ and every erase move $E(p, q, A, C)$, C is finite.

Except for the insertion of the adjective "generalized" in the obvious places, the concepts of stack-counter acceptors carry over to generalized stack-counter acceptors. (We omit the details.) In this way we get $\mathscr{L}_{\mathrm{QRG}}$, the family of quasi-realtime generalized stack-counter acceptor languages, and $\mathscr{L}_{\mathrm{BQRG}}$, the family of bounded quasi-realtime generalized stack-counter acceptor languages.

Turning to the third of the three new families, we have:

DEFINITION. Let $S = (M, q_0, q_1)$ be a stack-counter acceptor. A word w in Σ^* is said to be accepted in *SE-quasi-realtime* (*with delay d*) if $(q_0, w, \uparrow) \vdash_{\mu_1} \cdots \vdash_{\mu_n} (q_1, \epsilon, \uparrow)$ for some sequence of moves in M, $n \geqslant 0$, such that, among any consecutive ϵ-moves of the sequence, there are at most d moves other than store and erase moves. If each word in $L(S)$ is accepted in SE-quasi-realtime with delay d, then S is called an *SE-quasi-realtime stack-counter acceptor* (*with delay* d).

Thus $\mathscr{L}_{\text{SEQR}}$, the family of "$SE$-quasi-realtime stack-counter acceptor languages," becomes defined.

We are bow ready for the main result of this section.

PROPOSITION 3.1. $\mathscr{L}_{\text{QRG}} \subseteq \mathscr{L}_{\text{SEQR}}$ *and* $\mathscr{L}_{\text{BQRG}} \subseteq \mathscr{L}_{\text{QR}}$.

Proof. Let $S = (M, q_0, q_1)$ be a quasi-realtime generalized stack-counter acceptor and let d be the delay of S. Let K_0 and Σ_0 be the set of states and inputs respectively of S. Let

$$\begin{aligned}\mathscr{R} &= \{A, C/S(p, q, A, C) \text{ or } E(p, q, A, C) \text{ in } M \text{ for some } p, q\}\\ &\cup \{A, \{\gamma\}, B/R(p, q, A, \gamma, B) \text{ or } L(p, q, A, \gamma, B) \text{ in } M \text{ for some } p, q\}\\ &\cup \{\{\gamma\}, \{\alpha\}, B/\not c S(p, q, \gamma, \alpha, B) \text{ or } \not c E(p, q, \gamma, \alpha, B) \text{ in } M \text{ for some } p, q\}.\end{aligned}$$

(Thus $\mathscr{R}$ consists of all the subsets of c^* explicitly mentioned in the generalized moves of S.) Then $\mathscr{R}$ is a finite family of regular sets. For each R in $\mathscr{R}$ let $\equiv_R$ be a finite-index congruence relation as given by Proposition 2.1. Let $\equiv$ be the intersection of these congruences, i.e., $\alpha \equiv \beta$ if and only if $\alpha \equiv_R \beta$ for each R in $\mathscr{R}$. Clearly $\equiv$ is a finite-index congruence relation on c^* with the property that, for each R in $\mathscr{R}$, if $\alpha \equiv \beta$ and α is in R then β is in R. Let π be the period and κ the constant of $\equiv$.[9] For each word α in c^* let $\langle\alpha\rangle$ be the shortest word w satisfying $w \equiv \alpha$.

We now define $\bar{S} = (\bar{M}, q_0(\epsilon, \epsilon), q_1(\epsilon, \epsilon))$. Since $\bar{S}$ is to be SE-quasi-realtime, in any sequence of consecutive ϵ-moves all but a small number are to be stores and erases. (In addition, if $\bar{S}$ is bounded, then there are to be only a limited number of total consecutive ϵ-moves in S.) Thus, whenever we wish to simulate a $\not c S(p, q, \bar{\gamma}, \bar{\alpha}, B)$ move, we have to ensure that the right stack word in $\bar{S}$ is not too large (since the stack word can only be increased from the right end). We accomplish this by simulating the stack $\alpha\uparrow\beta$ in S by $\alpha/\beta \uparrow \beta^2$ in $\bar{S}$ if $\beta \leqslant \alpha$, and by $\beta/\alpha \uparrow \alpha^2$ if $\alpha < \beta$. We shall see that the delay of $\bar{S}$ is $9 \mid \pi\kappa \mid d$.

The set $\bar{K}$ of states of $\bar{S}$ consists of all symbols $p(\langle\alpha\rangle, \langle\beta\rangle)$, $\bar{p}(\langle\alpha\rangle, \langle\beta\rangle)$, $p_\mu(\langle\alpha\rangle, \langle\beta\rangle, \langle\gamma\rangle)$, $\bar{p}_\mu(\langle\alpha\rangle, \langle\beta\rangle, \langle\gamma\rangle)$, and finitely many other states as implied by the description of $\bar{M}$; for each p in K_0, μ in M, and α, β and γ in c^*. In general, the pair $(p, \alpha\uparrow\beta)$ is to be represented by $(p(\langle\alpha\rangle, \langle\beta\rangle), \alpha/\beta \uparrow \beta^2)$ if $\beta \leqslant \alpha$ and by $(\bar{p}(\langle\alpha\rangle, \langle\beta\rangle), \beta/\alpha \uparrow \alpha^2)$ if $\alpha < \beta$. In particular, the set $\bar{M}$ will be constructed so that, for each word w_0 in Σ^*,

$$(*)\quad \text{if } (q_0, w_0, \uparrow) \overset{*}{\underset{M}{\vdash}} (p, w, \alpha \uparrow \beta) \text{ then } (\alpha)\ (q_0(\epsilon, \epsilon), w_0, \uparrow) \overset{*}{\underset{\bar{M}}{\vdash}} (p(\langle\alpha\rangle, \langle\beta\rangle), w, \alpha/\beta \uparrow \beta^2)$$
$$\text{if } \beta \leqslant \alpha, \text{ and } (\beta)\ (q_0(\epsilon, \epsilon), w_0, \uparrow) \overset{*}{\underset{\bar{M}}{\vdash}} (\bar{p}(\langle\alpha\rangle, \langle\beta\rangle), w, \beta/\alpha \uparrow \alpha^2) \text{ if } \alpha < \beta.$$

[9] Note that, if R is a finite set in $\mathscr{R}$, then $\alpha < \kappa$ for every word α in R.

To define $\bar{M}$ it is convenient to first introduce certain subroutines (i.e., finite sets of moves). In each case, we briefly outline the moves in the subroutine. Each subroutine produces transformations between configurations with explicitly listed states. Aside from these states, the subroutines have no states in common with each other or with moves explicitly listed in $\bar{M}$ below.

There are four kinds of subroutines. These are as follows (for all states $\tilde{p}$, $\tilde{q}$ in $\bar{K}$, $\tilde{p} \neq \tilde{q}$, all words w in Σ^*, and all words $\alpha, \beta, \gamma, \rho$ in c^*):

(a) $\tilde{p} \xrightarrow[\alpha]{W} \tilde{q}$. In at most $2 \mid \alpha \mid + 1$ consecutive moves, this set of moves transforms $(\tilde{p}, w, \lambda\uparrow)$ into $(\tilde{q}, w, \lambda\uparrow)$ if and only if either $\alpha = \lambda$ or $\kappa \leqslant \alpha \leqslant \lambda$. (To implement, guess that $\alpha = \lambda$ or that $\kappa \leqslant \alpha \leqslant \lambda$. Since κ and α are given, it is known whether or not $\kappa \leqslant \alpha$. If the guess is that $\alpha = \lambda$, then use $\mid \alpha \mid$ erase moves, one left end move, and $\mid \alpha \mid$ store moves. If $\kappa \leqslant \alpha$ and the guess is that $\alpha \leqslant \lambda$, use $\mid \alpha \mid$ erase moves and $\mid \alpha \mid$ stores moves.)

(b) $\tilde{p} \xrightarrow[\beta]{RW} \tilde{q}$. In at most $4 \mid \beta \mid + 1$ consecutive moves, this set of moves transforms $(\tilde{p}, w, \lambda\uparrow\rho)$ into $(\tilde{q}, w, \lambda\uparrow\rho)$ if and only if either $\beta^2 = \rho$ or $\kappa^2 \leqslant \beta^2 \leqslant \rho$. (To implement, guess that $\beta^2 = \rho$ or $\kappa^2 \leqslant \beta^2$. Since κ and β are given, it is known whether or not $\kappa^2 \leqslant \beta^2$. If the guess is that $\beta^2 = \rho$, use $2 \mid \beta \mid$ right shifts, one right end move, and $2 \mid \beta \mid$ left shifts. If $\kappa^2 \leqslant \beta^2$ and the guess is that $\beta^2 \leqslant \rho$, use $2 \mid \beta \mid$ right shifts and $2 \mid \beta \mid$ left shifts.)

(c) $\tilde{p} \xrightarrow[\beta]{LW} \tilde{q}$. In at most $2 \mid \beta \mid + 1$ consecutive moves, this set transforms $(\tilde{p}, w, \lambda\uparrow\rho)$ into $(\tilde{q}, w, \lambda\uparrow\rho)$ if and only if either $\beta = \lambda\alpha$ or $\kappa \leqslant \beta \leqslant \lambda\alpha$. (To implement, guess that $\beta = \lambda\alpha$ or $\kappa \leqslant \beta \leqslant \lambda\alpha$. Obviously it is known whether or not $\kappa \leqslant \beta$. If the guess is that $\beta = \lambda\alpha$, use $\mid \beta \mid - \mid \alpha \mid$ left shifts, one left end move, and $\mid \beta \mid - \mid \alpha \mid$ right shifts. If $\kappa \leqslant \beta$ and the guess is that $\beta \leqslant \lambda\alpha$, use $\mid \beta \mid - \mid \alpha \mid$ left shifts and $\mid \beta \mid - \mid \alpha \mid$ right shifts.)

(d) $L(\tilde{p}, \tilde{q}, \alpha)$. In exactly $\mid \alpha \mid$ consecutive moves, this set transforms $(\tilde{p}, w, \lambda\uparrow)$ into $(\tilde{q}, w, \lambda/\alpha \uparrow \alpha)$ if and only $\alpha \leqslant \lambda$. (To implement, use $\mid \alpha \mid$ left shifts.)

Mnemonically, W stands for stack *w*ord, RW for *r*ight stack *w*ord, LW for *l*eft stack *w*ord, and L in $L(\tilde{p}, \tilde{q}, \alpha)$ for *l*eft on the stack.

Formally, $\bar{M}$ consists of the following moves and subroutines (for all μ in M and α, β, γ in c^*, with $\alpha < \pi\kappa$, $\beta < \pi\kappa$, and $\gamma < \pi\kappa$):

(1) For $\mu = I(p, q, a)$, $\bar{M}$ contains $I(p(\alpha, \beta), q(\alpha, \beta), a)$ and $I(\bar{p}(\alpha, \beta), \bar{q}(\alpha, \beta), a)$.

(2) For $\mu = S(p, q, A, C)$, $\bar{M}$ contains

(a) $\$(p(\alpha, \epsilon), p_\mu(\alpha, \epsilon, \epsilon))$ if α is in A,

(b) $S(p_\mu(\alpha, \epsilon, \gamma), p_\mu(\langle\alpha c\rangle, \epsilon, \langle\gamma c\rangle))$, and

(c) $p_\mu(\alpha, \epsilon, \gamma) \rightarrow q(\alpha, \epsilon)$ if γ is in C.

(Thus, if $(p, w, \bar{\alpha}\uparrow) \vdash_{\mu} (q, w, \bar{\alpha}\bar{\gamma}\uparrow)$ with $\bar{\alpha}$ in A and $\bar{\gamma}$ in C, then

$$(p(\langle\bar{\alpha}\rangle, \epsilon), w, \bar{\alpha}\uparrow) \overset{*}{\underset{\bar{M}}{\vdash}} (q(\langle\bar{\alpha}\rangle, \epsilon), w, \bar{\alpha}\bar{\gamma}\uparrow).$$

Clearly this can be done using exactly $2 + |\bar{\gamma}|$ ϵ-moves, all but two being stores.)

(3) For $\mu = E(p, q, A, C)$, $\bar{M}$ contains

(a) $\$(p(\alpha, \epsilon), p_{\mu}(\alpha, \epsilon, \epsilon))$,

(b) $E(p_{\mu}(\alpha, \epsilon, \gamma), p_{\mu}(\alpha/c, \epsilon, \langle\gamma c\rangle))$ and $E(p_{\mu}(\kappa, \epsilon, \gamma), p_{\mu}(\pi\kappa/c, \epsilon, \langle\gamma c\rangle))$, and

(c) $p_{\mu}(\alpha, \epsilon, \gamma) \xrightarrow[\alpha]{W} q(\alpha, \epsilon)$ if α is in A and γ in C.

(Thus, if $(p, w, \bar{\alpha}\uparrow) \vdash_{\mu} (q, w, \bar{\alpha}/\bar{\gamma}\uparrow)$, with $\bar{\alpha}/\bar{\gamma}$ in A and $\bar{\gamma}$ in C, then

$$(p(\langle\bar{\alpha}\rangle, \epsilon), w, \bar{\alpha}\uparrow) \overset{*}{\underset{\bar{M}}{\vdash}} (q(\langle\bar{\alpha}/\bar{\gamma}\rangle, \epsilon), w, \bar{\alpha}/\bar{\gamma}\uparrow).$$

Clearly this can be done in at most $1 + |\bar{\gamma}| + 2|\langle\bar{\alpha}_{\bar{\gamma}}\rangle| + 1 \leqslant |\bar{\gamma}| + 2|\pi\kappa|$ ϵ-moves, all but at most two different from stores and erases.)

(4) For $\mu = R(p, q, A, \bar{\gamma}, B)$, $\bar{M}$ contains

(a) $p(\alpha, \beta) \rightarrow p_{\mu}(\alpha, \beta, \epsilon)$ and $\bar{p}(\alpha, \beta) \rightarrow \bar{p}_{\mu}(\alpha, \beta, \epsilon)$, if α is in A,

(b) $R(p_{\mu}(\alpha, \beta, \gamma), p_{\mu}'(\langle\alpha c\rangle, \beta/c, \langle\gamma c\rangle))$ and $R(p_{\mu}(\alpha, \kappa, \gamma), p_{\mu}'\langle(\alpha c\rangle, \pi\kappa/c, \langle\gamma c\rangle))$,

(c) $R(p_{\mu}'(\alpha, \beta, \gamma), p_{\mu}(\alpha, \beta, \gamma))$,

(d) $L(\bar{p}_{\mu}(\alpha, \beta, \gamma), \bar{p}_{\mu}'(\langle\alpha c\rangle, \beta/c, \langle\gamma c\rangle))$ and $L(\bar{p}_{\mu}(\alpha, \kappa, \gamma), \bar{p}_{\mu}'(\langle\alpha c\rangle, \pi\kappa/c, \langle\gamma c\rangle))$,

(e) $L(\bar{p}_{\mu}'(\alpha, \beta, \gamma), \bar{p}_{\mu}(\alpha, \beta, \gamma))$,

(f) $\cent(\bar{p}_{\mu}(\alpha, \beta, \gamma), p_{\mu}(\alpha, \beta, \gamma))$ and $\cent(\bar{p}_{\mu}'(\alpha, \beta, \gamma), p_{\mu}'(\alpha, \beta, \gamma))$,

(g) $p_{\mu}(\alpha, \beta, \bar{\gamma}) \xrightarrow[\beta]{RW} q(\alpha, \beta)$ if β is in B, and

(h) $\bar{p}_{\mu}(\alpha, \beta, \bar{\gamma}) \xrightarrow[\beta]{LW,\alpha} \bar{q}(\alpha, \beta)$ if β is in B.

(Thus, if $(p, w, \bar{\alpha}\uparrow\bar{\beta}) \vdash_{M} (q, w, \bar{\alpha}\bar{\gamma}\uparrow \bar{\beta}/\bar{\gamma})$, with $\bar{\alpha}$ in A and $\bar{\beta}/\bar{\gamma}$ in B, then either

(i) $\bar{\beta} \leqslant \bar{\alpha}$ and $(p(\langle\bar{\alpha}\rangle, \langle\bar{\beta}\rangle), w, \bar{\alpha}/\bar{\beta} \uparrow \bar{\beta}^2) \vdash_{\bar{M}}^{*} (q(\langle\bar{\alpha}\bar{\gamma}\rangle, \langle\bar{\beta}/\bar{\gamma}\rangle), w, \bar{\alpha}\bar{\gamma}/(\bar{\beta}/\bar{\gamma}) \uparrow (\bar{\beta}/\bar{\gamma})^2)$,

or

(ii) $\bar{\alpha} < \bar{\beta}$, $\bar{\alpha}\bar{\gamma} < \bar{\beta}/\bar{\gamma}$, and

$$(\bar{p}(\langle\bar{\alpha}\rangle, \langle\bar{\beta}\rangle), w, \bar{\beta}/\bar{\alpha} \uparrow \bar{\alpha}^2) \vdash_{\bar{M}}^{*} (q(\langle\bar{\alpha}\bar{\gamma}\rangle, \langle\bar{\beta}/\bar{\gamma}\rangle), w, (\bar{\beta}/\bar{\gamma})/\bar{\alpha}\bar{\gamma} \uparrow (\bar{\alpha}\bar{\gamma})^2),$$

or

(iii) $\bar{\alpha} < \bar{\beta}$, $\bar{\alpha}\bar{\gamma} \geqslant \bar{\beta}/\bar{\gamma}$, and

$$(\bar{p}(\langle\bar{\alpha}\rangle, \langle\bar{\beta}\rangle), w, \bar{\beta}/\bar{\alpha} \uparrow \bar{\alpha}^2) \vdash_{\bar{M}}^{*} (q(\langle\bar{\alpha}\bar{\gamma}\rangle, \langle\bar{\beta}/\bar{\gamma}\rangle), w, \bar{\alpha}\bar{\gamma}/(\bar{\beta}/\bar{\gamma}) \uparrow (\bar{\beta}/\bar{\gamma})^2).$$

(Case (i) can be done in at most $1 + 2 \mid \bar{\gamma} \mid + 4 \mid \langle \bar{\beta}_{\bar{\gamma}} \rangle \mid + 1 \leqslant 6 \mid \pi\kappa \mid$ ϵ-moves, Case (ii) in at most $1 + 2\bar{\gamma} + 2\langle \bar{\beta}_{\bar{\gamma}} \rangle + 1 \leqslant 4\pi\kappa$ ϵ-moves, and Case (iii) in at most $6\pi\kappa$ ϵ-moves.)

(5) For $\mu = L(p, q, A, \bar{\gamma}, B)$, $\bar{M}$ contains

(a) $p(\alpha, \beta) \rightarrow p_\mu(\alpha, \beta, \epsilon)$ and $\bar{p}(\alpha, \beta) \rightarrow \bar{p}_\mu(\alpha, \beta, \epsilon)$ if β is in B,

(b) $L(p_\mu(\alpha, \beta, \gamma), p_\mu'(\alpha/c, \langle\beta c\rangle, \langle\gamma c\rangle))$ and $L(p_\mu(\kappa, \beta, \gamma), p_\mu'(\kappa\pi/c, \langle\beta c\rangle, \langle\gamma c\rangle))$,

(c) $L(p_\mu'(\alpha, \beta, \gamma), p_\mu(\alpha, \beta, \gamma))$,

(d) $R(\bar{p}_\mu(\alpha, \beta, \gamma), \bar{p}_\mu'(\alpha/c, \langle\beta c\rangle, \langle\gamma c\rangle))$ and $R(\bar{p}_\mu(\kappa, \beta, \gamma), \bar{p}_\mu'(\kappa\pi/c, \langle\beta c\rangle, \langle\gamma c\rangle))$,

(e) $R(\bar{p}_\mu'(\alpha, \beta, \gamma), \bar{p}_\mu(\alpha, \beta, \gamma))$,

(f) $\not{c}(p_\mu(\alpha, \beta, \gamma), \bar{p}_\mu(\alpha, \beta, \gamma))$ and $\not{c}(p_\mu'(\alpha, \beta, \gamma), \bar{p}_\mu'(\alpha, \beta, \gamma))$,

(g) $p_\mu(\alpha, \beta, \bar{\gamma}) \xrightarrow[\alpha]{LW,\beta} q(\alpha, \beta)$ if α is in A, and

(h) $\bar{p}_\mu(\alpha, \beta, \bar{\gamma}) \xrightarrow[\alpha]{RW} \bar{q}(\alpha, \beta)$ if α is in A.

(This is analogous to (4).)

(6) For $\mu = \not{c}S(p, q, \bar{\gamma}, \bar{\alpha}, B)$, $\bar{\beta}$ in B, $\bar{\beta} \leqslant (\pi\kappa)^2$, $\bar{M}$ contains[10]

(a) $p(\bar{\alpha}, \langle\bar{\beta}\rangle) \rightarrow p_\mu(\bar{\alpha}, \bar{\beta}, \epsilon)$ and $\bar{p}(\bar{\alpha}, \langle\bar{\beta}\rangle) \rightarrow \bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \epsilon)$,

(b) $R(p_\mu(\bar{\alpha}, \bar{\beta}, \epsilon), p_\mu(\bar{\alpha}, \bar{\beta}, \epsilon))$ and $R(\bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \epsilon), \bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \epsilon))$,

(c) $S(p_\mu(\bar{\alpha}, \bar{\beta}, \gamma), p_\mu(\bar{\alpha}, \bar{\beta}, \langle\gamma c\rangle))$ and $S(\bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \gamma), \bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \langle\gamma c\rangle))$,

(d) $L(p_\mu(\bar{\alpha}, \bar{\beta}, \bar{\gamma}), q(\langle\bar{\gamma}\bar{\alpha}\rangle, \langle\bar{\beta}\rangle), \bar{\beta}^2)$,

(e) $L(\bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \bar{\gamma}), \bar{q}(\langle\bar{\gamma}\bar{\alpha}\rangle), \langle\bar{\beta}\rangle, (\bar{\gamma}\bar{\alpha})^2)$, and

(f) $L(\bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \bar{\gamma}), q(\langle\bar{\gamma}\bar{\alpha}\rangle, \langle\bar{\beta}\rangle), \bar{\beta}^2)$.

(Thus, if $(p, w, \bar{\alpha}\uparrow\bar{\beta}) \vdash_\mu (q, w, \bar{\gamma}\bar{\alpha}\uparrow\bar{\beta})$, with $\bar{\beta}$ in B, then either

(i) $\bar{\beta} \leqslant \bar{\alpha}$ and $(p(\langle\bar{\alpha}\rangle, \langle\bar{\beta}\rangle), w, \bar{\alpha}/\bar{\beta} \uparrow \bar{\beta}^2) \overset{*}{\vdash}_{\bar{M}} (q(\langle\bar{\gamma}\bar{\alpha}\rangle, \langle\bar{\beta}\rangle), w, \bar{\gamma}\bar{\alpha}/\bar{\beta} \uparrow \bar{\beta}^2)$, or

(ii) $\bar{\alpha} < \bar{\beta}, \bar{\gamma}\bar{\alpha} < \bar{\beta}$, and $(\bar{p}(\langle\bar{\alpha}\rangle, \langle\bar{\beta}\rangle, w, \bar{\beta}/\bar{\alpha} \uparrow \bar{\alpha}^2) \overset{*}{\vdash}_{\bar{M}} (\bar{q}(\langle\bar{\gamma}\bar{\alpha}\rangle, \langle\bar{\beta}\rangle), w, \bar{\beta}/\bar{\gamma}\bar{\alpha} \uparrow (\bar{\gamma}\bar{\alpha})^2)$,

or

(iii) $\bar{\alpha} < \bar{\beta}, \bar{\beta} \leqslant \bar{\gamma}\alpha$, and $(\bar{p}(\langle\bar{\alpha}\rangle, \langle\bar{\beta}\rangle), w, \bar{\beta}/\bar{\alpha} \uparrow \bar{\alpha}^2) \overset{*}{\vdash}_{\bar{M}} (q(\langle\bar{\gamma}\bar{\alpha}\rangle, \langle\bar{\beta}\rangle, w, \bar{\gamma}\bar{\alpha}/\bar{\beta} \uparrow \bar{\beta}^2)$.

(It is easily seen that each case can be done in at most $7 \mid \pi\kappa \mid$ ϵ-moves.)

(7) For $\mu = \not{c}E(p, q, \bar{\gamma}, \bar{\alpha}, B)$, $\bar{\beta}$ in B, $\bar{\beta} \leqslant (\pi\kappa)^2$, $\bar{M}$ contains

(a) $p(\langle\bar{\gamma}\bar{\alpha}\rangle, \langle\bar{\beta}\rangle) \rightarrow p_\mu(\bar{\alpha}, \bar{\beta}, \epsilon)$ and $\bar{p}(\langle\bar{\gamma}\bar{\alpha}\rangle, \langle\bar{\beta}\rangle) \rightarrow \bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \epsilon)$,

(b) $R(p_\mu(\bar{\alpha}, \bar{\beta}, \epsilon), p_\mu(\bar{\alpha}, \bar{\beta}, \epsilon))$ and $R(\bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \epsilon)), \bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \epsilon))$,

[10] See footnote 9.

(c) $E(p_\mu(\bar{\alpha}, \bar{\beta}, \gamma), p_\mu(\bar{\alpha}, \bar{\beta}, \langle\gamma c\rangle))$ and $E(\bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \gamma), p_\mu(\bar{\alpha}, \bar{\beta}, \langle\gamma c\rangle))$,

(d) $L(p_\mu(\bar{\alpha}, \bar{\beta}, \bar{\gamma}), q(\bar{\alpha}, \langle\bar{\beta}\rangle), \bar{\beta}^2)$,

(e) $L(\bar{p}_\mu(\bar{\alpha}, \bar{\beta}, \bar{\gamma}), \bar{q}(\bar{\alpha}, \langle\bar{\beta}\rangle), \bar{\alpha}^2)$, and

(f) $L(p_\mu(\bar{\alpha}, \bar{\beta}, \bar{\gamma}), \bar{q}(\bar{\alpha}, \langle\bar{\beta}\rangle), \bar{\alpha}^2)$.

(This is analogous to (6). There are at most $9 \mid \pi\kappa \mid$ ϵ-moves.)

Thus (*) holds, and each ϵ-move of S is simulated by at most $9 \mid \pi\kappa \mid$ ϵ-moves of $\bar{S}$ other than stores and erases. If S is bounded, then, by the construction, each ϵ-move of S is simulated by at most $9 \mid \pi\kappa \mid$ total ϵ-moves of $\bar{S}$.

To complete the argument of Proposition 3.1, it is straightforward (proof omitted) to show the following:

(**) Suppose that $\hat{\xi} \vdash\!\!\frac{*}{M}\, \hat{\eta}$, where $\hat{\xi} = (p(\langle\alpha\rangle, \langle\beta\rangle), w, \alpha/\beta \uparrow \beta^2)$ or $\hat{\xi} = (\bar{p}(\langle\alpha\rangle, \langle\beta\rangle, w, \beta/\alpha \uparrow \alpha^2)$, p in K_0, and $\hat{\eta}$ has state $q(s, t)$ or $\bar{q}(s, t)$, q in K_0. Then either $\hat{\eta} = (q(\langle\gamma\rangle, \langle\delta\rangle, w', \gamma/\delta \uparrow \delta^2)$ or $\hat{\eta} = (\bar{q}(\langle\gamma\rangle, \langle\delta\rangle), w'(\delta/\gamma) \uparrow \gamma^2)$, where $(p, w, \alpha\uparrow\beta) \vdash\!\!\frac{*}{M} (q, w', \gamma\uparrow\delta)$.

Remarks. The proof of Proposition 3.1 also shows that each generalized stack-counter language is a stack-counter language.

4. $\mathscr{L} \subseteq \mathscr{L}_{\mathrm{QRG}}$

The purpose of the present section is to prove the second of the three main auxiliary results, namely,

PROPOSITION 4.1. $\mathscr{L} \subseteq \mathscr{L}_{\mathrm{QRG}}$.

Proof. Let $S = (M, q_0, q_1)$ be an arbitrary stack-counter acceptor. We shall exhibit a quasi-realtime generalized stack-counter acceptor $\bar{S} = (\bar{M}, q_0, q_1)$ such that $L(\bar{S}) = L(S)$.

Let $\equiv_S$ (abbreviated $\equiv$) be a finite-index congruence relation on c^* with the properties guaranteed by Proposition 2.4. As in Section 3, for each word α in c^* let $\langle\alpha\rangle$ be the shortest word in $[\alpha]$. Let π be the period and κ the constant of $\equiv$. Let $\Delta = \pi^{2|\pi\kappa|}$.

We now summarize some simple facts, each to be implicity used in the sequel, about the above concepts. Each follows readily from Proposition 2.2.

LEMMA 4.1. (a) $\Delta > (\pi k)^2$.

(b) *Δ is in π^*.*

(c) *If $\alpha \equiv \beta$ and $\alpha \leqslant \beta$ then β/α is in π^*.*

(d) *If* $\kappa \leqslant \alpha$, *then* $[\alpha]\,\pi^* = [\alpha]$.

(e) *If* $\beta \equiv \gamma$ *and* $\alpha \geqslant \beta$, *then* $(\alpha/\beta)\gamma \equiv \alpha$.

(f) *If* $\beta \equiv \gamma$, $\kappa \leqslant \alpha$, and $\kappa \leqslant \alpha\beta/\gamma$, *then* $\alpha\beta/\gamma \equiv \alpha$.

For each state p in K_0 let $\bar{p}$, p', and $\bar{p}'$ be new symbols. Let $\bar{K}_0 = \{p, \bar{p}, p', \bar{p}'/p \text{ in } K_0\}$ be the set of states of $\bar{S}$. The roles of the states are to be as follows. The configuration $(p, w, \alpha/\beta \uparrow \beta^2)$ of $\bar{S}$ simulates $(p, w, \alpha\uparrow\beta)$ of S when $\alpha \geqslant \beta$. The configuration $(\bar{p}, w, \beta/\alpha \uparrow \alpha^2)$ of $\bar{S}$ simulates $(p, w, \alpha\uparrow\beta)$ of S when $\alpha < \beta$. The configuration $(p', w, \alpha/\beta \uparrow \beta^2)$ of $\bar{S}$ simulates $(p, w, \alpha\uparrow\beta)$ of S when $\alpha \geqslant \beta$ and *every* configuration $(p, w, \alpha'\uparrow\beta')$, with $\alpha' \equiv \alpha$, $\beta' \equiv \beta$, $\alpha'\beta' = \alpha\beta$, and $\beta \leqslant \min(\alpha', \beta')$, is obtainable in S from the start configuration of S. The configuration $(\bar{p}', w, \beta/\alpha \uparrow \alpha^2)$ of $\bar{S}$ simulates $(p, w, \alpha\uparrow\beta)$ of S when $\alpha < \beta$ and *some* configuration $(p, w, \alpha'\uparrow\beta')$, with $\alpha \equiv \alpha'$, $\beta \equiv \beta'$, $\alpha'\beta' = \alpha\beta$, and $\alpha \leqslant \min(\alpha', \beta')$, is obtainable in S from the start configuration of S.

The generalized moves of $\bar{M}$ are next given. The reader is avised to omit the moves on a first reading and return to them as they are later used. $\bar{M}$ consists of the generalized moves (1)–(6) (for all states p, q, in K_0 and all words α, β, γ in c^*, with $\alpha < \Delta^8$, $\beta < \Delta^8$, and $\gamma < \Delta^8$):

(1) If $I(p, q, a)$ is in M, then $\bar{M}$ contains $I(p, q, a)$, $I(\bar{p}, \bar{q}, a)$, $I(p', q', a)$, and $I(\bar{p}', \bar{q}', a)$.

(2) *If* $(p, w, \alpha\uparrow) \vdash_M^* (q, w, \alpha\gamma\uparrow)$ over α, then $\bar{M}$ contains $S(p, q, [\alpha], [\gamma])$.

(3) If $(p, w, \alpha\gamma\uparrow) \vdash_M^* (q, w, \alpha\uparrow)$ over α, then $\bar{M}$ contains $E(p, w, [\alpha], [\gamma])$.

(4) *If* $(p, w, \alpha\uparrow\gamma\beta) \vdash_M^* (q, w, \alpha\gamma\uparrow\beta)$ over $\alpha\gamma\beta$, then $\bar{M}$ contains:

(a) $R\left(p, q, \dfrac{[\alpha]}{\gamma[\beta]}, \gamma^2, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\}\right)$,

(b) $R\left(\bar{p}, q, \dfrac{[\gamma]\beta}{\alpha}, \dfrac{\alpha^2}{\beta^2}, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\}\right)$,[11]

(c) $L\left(\bar{p}, q, \dfrac{\alpha[\gamma]}{\beta}, \dfrac{\beta^2}{\alpha^2}, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\}\right)$,

(d) $L\left(\bar{p}, \bar{q}, \dfrac{[\beta]}{[\alpha]\gamma}, \gamma^2, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\}\right)$,

(e) $L\left(\bar{p}, q', \dfrac{\alpha[\gamma]}{\beta}, \dfrac{\beta^2}{\alpha^2}, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\}\right)$, if $\pi\kappa \leqslant \dfrac{\beta}{\alpha}$ and $\kappa \leqslant \gamma$,

(f) $L\left(p', q', \dfrac{[\alpha]\langle\gamma\rangle}{[\beta]}, \dfrac{\Delta^4}{\langle\gamma\rangle^2}, \{\bar{\beta}^2/\bar{\beta} \equiv \gamma\beta\}\right)$, if $\kappa \leqslant \alpha$ and $\kappa \leqslant \beta$,

(g) $R\left(\bar{p}', q, \dfrac{[\gamma]\beta}{\alpha}, \dfrac{\alpha^2}{\beta^2}, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\}\right)$, if $\pi\kappa \leqslant \dfrac{\alpha}{\beta}$ and $\kappa \leqslant \gamma$, and

(h) $R\left(\bar{p}', \bar{q}', \dfrac{\langle\gamma\rangle[\beta]}{[\alpha]}, \dfrac{\Delta^4}{\langle\gamma\rangle^2}, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\gamma\}\right)$, if $\kappa \leqslant \alpha$ and $\kappa \leqslant \beta$.

[11] Note that (b) is undefined if $\beta > \alpha$, and (c) is undefined if $\alpha > \beta$.

(5) If $(p, w, \alpha\gamma{\uparrow}\beta) \vdash_M^* (q, w, \alpha{\uparrow}\gamma\beta)$ over $\alpha\gamma\beta$, then $\overline{M}$ contains

(a) $L(p, q, \dfrac{[\alpha]}{\gamma[\beta]}, \gamma^2, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\})$,

(b) $L(p, \bar{q}, \dfrac{[\gamma]\beta}{\alpha}, \dfrac{\alpha^2}{\beta^2}, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\})$,

(c) $R(p, \bar{q}, \dfrac{\alpha[\gamma]}{\beta}, \dfrac{\beta^2}{\alpha^2}, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\})$,

(d) $L(p, q', \dfrac{[\alpha]}{\gamma[\beta]}, \gamma^2, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\})$, if $\kappa \leqslant \gamma$ and $\kappa \leqslant \alpha$,

(e) $R(\bar{p}, \bar{q}, \dfrac{[\beta]}{[\alpha]\gamma}, \gamma^2, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\})$,

(f) $L(p', q', \dfrac{[\alpha]}{\gamma[\beta]}, \gamma^2, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\})$,

(g) $R(\bar{p}', \bar{q}, \dfrac{[\beta]}{[\alpha]\gamma}, \gamma^2, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\})$, if $\kappa \leqslant \gamma$ and $\kappa \leqslant \alpha$, and

(h) $R(\bar{p}', \bar{q}', \dfrac{[\beta]}{[\alpha]\gamma}, \gamma^2, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\})$.

(6) $\overline{M}$ contains

(a) $R(q, \bar{q}', \dfrac{[\alpha]}{[\beta]}, \dfrac{\beta^2}{\alpha^2}, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\})$, if $\kappa \leqslant \alpha$,

(b) $R(\bar{q}, \bar{q}', \dfrac{[\beta]}{[\alpha]}, \gamma^2, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\})$, if $\alpha\gamma \equiv \alpha$,

(c) $R(q', \bar{q}', \dfrac{[\alpha]}{[\beta]}, \beta^2/\alpha^2, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\})$, if $\kappa \leqslant \alpha$,

(d) $L(p', p, \dfrac{[\alpha]}{[\beta]}, \gamma^2, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\})$, if $\gamma\beta \equiv \beta$, and

(e) $L(p', \bar{p}, \dfrac{[\beta]}{[\alpha]}, \dfrac{\alpha^2}{\beta^2}, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\})$, if $\kappa \leqslant \beta$.

Note that each set $[\alpha]$ is regular by Proposition 2.1. Since $\{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\}$ is the homomorphic image of $[\alpha]$ under the homomorphism which maps c into c^2, $\{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\}$ is regular. Thus $\bar{S} = (\overline{M}, q_0, q_1)$ is a generalized stack-counter acceptor.

We now show that $L(\bar{S}) = L(S)$ and that $\bar{S}$ never uses more than eight consecutive ϵ-moves. To see this we need some terminology for discussing the simulation of S by $\bar{S}$.

An element in $K_0 \times \Sigma^* \times (c^*\uparrow c^*)(\bar{K}_0 \times \Sigma^* \times (c^*\uparrow c^*))$ is called an *S-configuration* (*$\bar{S}$-configuration*). The S-configurations $\xi = (p, w, \alpha\uparrow\beta)$ and $\xi' = (p, w, \alpha'\uparrow\beta')$ are called *equivalent* (written $\xi \equiv \xi'$) if $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$.

For each configuration $\xi = (p, w, \alpha\uparrow\beta)$, call[12] $\min(\alpha, \beta)$ *the minimal stack word* of ξ. For configurations $\xi = (p_1, w, \alpha\uparrow\beta)$ and $\xi' = (p_2, w', \alpha'\uparrow\beta')$, with $\alpha\beta = \alpha'\beta'$,

(i) call $\max(|\min(\alpha, \beta)| - |\min(\alpha', \beta')|, |\min(\alpha', \beta')| - |\min(\alpha, \beta)|)$ the *difference* (abbreviated $|\xi, \xi'|$) between ξ and ξ', and

(ii) write $\xi \leqslant \xi'$ (or $\xi' \geqslant \xi$) if $\min(\alpha, \beta) \leqslant \min(\alpha', \beta')$.

For each $\bar{S}$-configuration ξ of the form $(p, w, \alpha/\beta \uparrow \beta^2)$, $(\bar{p}, w, \beta/\alpha \uparrow \alpha^2)$, $(p', w, \alpha/\beta \uparrow \beta^2)$, or $(\bar{p}', w, \beta/\alpha \uparrow \alpha^2)$, call $(p, w, \alpha\uparrow\beta)$, denoted by ξ^S, the *corresponding S-configuration.*

By a *simulator* of an S-configuration $\xi = (p, w, \alpha\uparrow\beta)$ is meant any $\bar{S}$-configuration ξ_1 of the form $(p, w, \alpha/\beta \uparrow \beta^2)$, or $(\bar{p}, w, \beta/\alpha \uparrow \alpha^2)$ with $\alpha < \beta$, or $(p', w, \alpha\beta/\beta_1^2 \uparrow \beta_1^2)$ $\kappa \leqslant \beta_1$, $\xi_1^S \equiv \xi$, and $\xi_1^S \leqslant \xi$.

Thus, if $(p', w, \alpha\beta/\beta_1^2 \uparrow \beta_1^2)$ is a simulator of $(p, w, \alpha\uparrow\beta)$, then $\alpha\beta/\beta_1 \equiv \alpha$ and $\beta_1 \equiv \beta$.

Let ξ_1 be any $\bar{S}$-configuration of the form $(p, w, \alpha/\beta \uparrow \beta^2)$, $(\bar{p}, w, \beta/\alpha \uparrow \alpha^2)$, or $(p', w, \alpha/\beta \uparrow \beta^2)$. By a *by-product* of ξ_1 is meant any $\bar{S}$-configuration ξ_2 either $= \xi_1$ or of the form $(\bar{p}', w, \alpha\beta/\alpha_2^2 \uparrow \alpha_2^2)$, with $\kappa \leqslant \alpha_2$, $\xi_2^S \equiv \xi_1^S$, and $\xi_2^S \leqslant \xi_1^S$.

In particular, suppose that $\xi = (p, w, \alpha\uparrow)$. Then the only simulators ξ_1 of ξ are of the form $(p, w, \alpha\uparrow)$ or $(p', w, \alpha\beta/\beta_1^2 \uparrow \beta_1^2)$, with $\xi_1^S \equiv \xi$ and $\kappa \leqslant \beta_1$. Hence ξ_1^S is either $(p, w, \alpha\uparrow)$ or $(p, w, \alpha\beta/\beta_1 \uparrow \beta_1)$. If ξ_1^S is the latter, then $\xi_1^S \equiv \xi$ implies $\beta_1 \equiv \epsilon$. Since $\epsilon < \kappa \leqslant \beta_1$, $\beta_1 \neq \epsilon$. Thus the only simulator of ξ is $(p, w, \alpha\uparrow)$. Similarly, the only by-product of $\eta = (p, w, \alpha\uparrow)$ or $\eta = (p', w, \alpha\uparrow)$ is η itself.

Informally speaking, $\bar{S}$ is constructed so that it simulates S, never using more than eight consecutive generalized ϵ-moves. When S, starting from $(q_0, w_0, \uparrow)$ reaches $\xi = (p, w, \alpha\uparrow\beta)$ just after making an input move, $\bar{S}$ will be in some simulator ξ_1 of ξ just after making the same input move. ξ_1^S will be in one of two cases:

(a) $\xi_1^S = \xi$.

(b) ξ_1^S is equivalent to ξ and has its right stack word shorter than both α and β. In this case, represented by the primed state, S can reach every S-configuration equivalent to and $\geqslant \xi_1^S$.

Suppose that $\bar{S}$ has reached a suitable simulator ξ_1 and that S makes another sequence of ϵ-moves followed by an input move, arriving at η. It could happen that there is no simulator for η which $\bar{S}$ can reach from ξ_1. However,

[12] For all words α and β in c^*, $\min(\alpha, \beta)$ denotes the shorter of the words α and β.

(*) among all the configurations $\bar{S}$ could have reached (after the same number of input moves as S) are the by-products of ξ_1, and $\bar{S}$ can reach a simulator η_1 of η from one such by-product ξ_2.

In order to transmit property (*) from input move to input move, $\bar{S}$ picks η_1 so that each of its by-products is accessible from at least one of the by-products of ξ_1. In addition to (*), $\bar{S}$ has the property that on entering ξ_2 some S-configuration which S could reach is equivalent to and $\geqslant \xi_2^S$. This is so even if S cannot reach ξ_2^S.

We now turn to formally showing that $\bar{S}$ has the desired properties of Proposition 4.1. We shall accomplish this with three lemmas. We first prove a technical result (Lemma 4.2). Then we show that $L(S) \subseteq L(\bar{S})$, with each word in $L(S)$ accepted by $\bar{S}$ with delay eight (Lemma 4.3). Finally we show that $L(\bar{S}) \subseteq L(S)$ (Lemma 4.4). The latter two lemmas combined yield Proposition 4.1.

We now consider the first of the three lemmas.

LEMMA 4.2. *Let $\xi \vdash_M^* \eta$ over $\alpha\gamma\beta$, where $\xi = (p, w, \alpha{\uparrow}\gamma\beta)$ and $\eta = (q, w, \alpha\gamma{\uparrow}\beta)$ ($\xi = (p, w, \alpha\gamma{\uparrow}\beta)$ and $\eta = (q, w, \alpha{\uparrow}\gamma\beta)$). Then for every simulator ξ_1 of ξ, there is a simulator η_1 of η with the following property: For each by-product η_2 of η_1, there is a by-product ξ_2 of ξ_1 for which $\xi_2 \vdash_{\bar{M}}^{d} \eta_2$ with $d \leqslant 3$.*

Proof. We shall prove only the case in which $\xi = (p, w, \alpha{\uparrow}\gamma\beta)$ and $\eta = (q, w, \alpha\gamma{\uparrow}\beta)$, the proof of the other case being analogous and using (5a)–(5h).

Let ξ_1 be a simulator of ξ. We first exhibit a simulator η_1 of η and a by-product ξ_2 of ξ_1 for which

(*) $$\xi_2 \vdash_{\bar{M}}^{m} \eta_1, \text{ with } m \leqslant 2 \text{ and } | \xi_2^S, \eta_1^S | < | \Delta^3 |.$$

A number of situations arise, the details of all but one of them (Case 2) being omitted.

Case 1. ξ_1 is $(p, w, \alpha/\gamma\beta \uparrow (\gamma\beta)^2)$. Suppose $| \xi, \eta | < | \Delta^2 |$. Then $(q, w, \alpha\gamma/\beta \uparrow \beta^2)$ is a suitable η_1, and ξ_1 serves as ξ_2. If $| \xi, \eta | \geqslant | \Delta^2 |$, then $(q, w, \alpha\gamma/\beta \uparrow \beta^2)$ is a suitable η_1, and $(\bar{p}', w, \alpha\gamma\beta/\alpha_2^2 \uparrow \alpha_2^2)$ serves as ξ_2, where $\alpha_2 = \Delta\langle\alpha\rangle\beta/\langle\beta\rangle$.

Case 2. ξ_1 is $(\bar{p}, w, \gamma\beta/\alpha \uparrow \alpha^2)$, with $\beta \leqslant \alpha$. If $| \xi, \eta | < | \Delta^2 |$, then $(q, w, \alpha\gamma/\beta \uparrow \beta^2)$ is a suitable η_1 and ξ_1 serves as ξ_2. (Here (4b) is used.) If $| \xi, \eta | \geqslant | \Delta^2 |$, then $(q, w, \alpha\gamma/\beta \uparrow \beta^2)$ is a suitable η_1, and $(\bar{p}', w, \alpha\gamma\beta/\alpha_2^2 \uparrow \alpha_2^2)$ serves as ξ_2, with $\alpha_2 = \Delta\langle\alpha\rangle\beta/\langle\beta\rangle$. [For $| \xi_2^S, \eta_1^S | = | \alpha_2/\beta | \leqslant | \Delta^2 |$. Since $| \alpha | \geqslant | \alpha/\beta | = | \xi, \eta | \geqslant | \Delta^2 |$ and $\gamma \geqslant \alpha/\beta \geqslant \kappa$, $\alpha_2 \equiv \alpha$ and $\alpha\gamma\beta/\alpha_2 = \alpha\gamma\langle\beta\rangle/\Delta\langle\alpha\rangle \equiv \gamma\beta$. Now $\alpha_2^2 \leqslant \Delta^4\beta^2 \leqslant (\alpha/\beta)\beta^2 \leqslant \alpha^2 \leqslant \alpha\gamma\beta$. Thus ξ_2 is a by-product of ξ_1. Then $\xi_2 \vdash_\mu \eta_1$, where

$$\begin{aligned}\mu &= R\left(\bar{p}', q, \frac{[\gamma]\beta}{\alpha_2}, \frac{\alpha_2^2}{\beta^2}, \{\bar{\beta}^2/\bar{\beta} \equiv \beta\}\right)\\ &= R\left(\bar{p}', q, \frac{[\langle\gamma\rangle]\langle\beta\rangle}{\Delta\langle\alpha\rangle}, \left(\frac{\Delta\langle\alpha\rangle}{\langle\beta\rangle}\right)^2, \{\bar{\beta}^2/\bar{\beta} \equiv \langle\beta\rangle\}\right)\end{aligned}$$

is in $\bar{M}$ by (4g).]

Case 3. ξ_1 in $(\bar{p}, w, \gamma\beta/\alpha \uparrow \alpha^2)$, with $\alpha < \beta \leqslant \alpha\gamma$. If $|\xi, \eta| < |\Delta^2|$, then $(q, w, \alpha\gamma/\beta \uparrow \beta^2)$ is a suitable η_1 and ξ_1 serves as ξ_2. If $|\xi, \eta| \geqslant |\Delta^2|$, then $(q', w, \alpha\gamma\beta/\beta_2^2 \uparrow \beta_2^2)$ is a suitable η_1, where $\xi_2 = \Delta\alpha\langle\beta\rangle/\langle\alpha\rangle$, and ξ_1 serves as ξ_2.

Case 4. ξ_1 is $(\bar{p}, w, \gamma\beta/\alpha \uparrow \alpha^2)$, with $\alpha\gamma < \beta$. If $|\xi, \eta| < |\Delta^2|$, then

$$(\bar{q}, w, \beta/\alpha\gamma \uparrow (\alpha\gamma)^2)$$

is a suitable η_1, and ξ_1 serves as ξ_2. If $|\xi, \eta| \geqslant |\Delta^2|$, then $(q', w, \alpha\gamma\beta/\beta_2^2 \uparrow \beta_2^2)$ is a suitable η_1, where $\beta_2 = \Delta\alpha\langle\beta\rangle/\langle\alpha\rangle$, and ξ_1 serves as ξ_2.

Case 5. ξ_1 has state p'. Then $\xi_1 = (p', w, \alpha_1^2/\alpha\gamma\beta \uparrow (\alpha\gamma\beta/\alpha_1)^2)$, where $\alpha_1 \equiv \alpha$, $\alpha\gamma\beta/\alpha_1 \equiv \gamma\beta$, and $\kappa \leqslant \alpha\gamma\beta/\alpha_1 \leqslant \min(\alpha, \gamma\beta)$. Let $\xi' = (p, w, \alpha\gamma/\langle\gamma\rangle \uparrow \langle\gamma\rangle\beta)$, and let ξ_1' denote the simulator of ξ' with state p or p'. Let η_0 denote the simulator of η with state q or $\bar{q}$.

First, suppose that $|\xi_1^S, \xi'| < |\Delta^2|$. Then η_0 is a suitable η_1, and ξ_1 serves as ξ_2. Suppose that $|\xi_1^S, \xi'| \geqslant |\Delta^2|$ and $\alpha_1 \leqslant \alpha\gamma/\langle\gamma\rangle$. Then η_0 is a suitable η_1 and $(\bar{p}, w, \alpha\gamma\beta/\alpha_2^2 \uparrow \alpha_2^2)$ serves as ξ_2, where $\alpha_2 = \Delta\langle\alpha\rangle\beta/\langle\beta\rangle$. Finally, suppose that $|\xi_1^S, \xi'| \geqslant |\Delta^2|$ and $\alpha_1 > \alpha\gamma/\langle\gamma\rangle$. Then $\eta_1 = (q', w, \alpha\gamma\beta/\beta_1^2 \uparrow \beta_1^2)$ is a simulator of η, where $\beta_1 = \Delta^2\alpha\gamma\beta/\langle\gamma\rangle\alpha_1$, and ξ_1 serves as ξ_2.

Thus (*) is established.

Now consider any by-product η_2 of η_1, $\eta_2 \neq \eta_1$. Observe that

$$\eta_2 = \left(\bar{q}', w, \frac{\beta_2^2}{\alpha\gamma\beta} \uparrow \left(\frac{\alpha\gamma\beta}{\beta_2}\right)^2\right),$$

$\kappa \leqslant \alpha\gamma\beta/\beta_2 =$ minimum stack word of $\eta_2 \leqslant$ minimum stack word of η_1, $\alpha\gamma\beta/\beta_2 \equiv \alpha\gamma$, and $\beta_2 \equiv \beta$. We shall exhibit a by-product ξ_3 of ξ_1 such that $\xi_3 \overset{d}{\vdash}_M \eta_2$, with $d \leqslant 3$. Two alternatives arise:

(α) $|\eta_1^S, \eta_2^S| < |\Delta^5|$. Then ξ_2 serves as ξ_3. To see this suppose that $\eta_1 = (q, w, \alpha\gamma\beta/\beta_1^2 \uparrow \beta_1^2)$. Then $\alpha\gamma\beta/\beta_2 \leqslant \beta_1 \leqslant \alpha\gamma\beta/\beta_1 \leqslant \beta_2$, $\alpha\gamma\beta/\beta_2 \equiv \alpha\gamma\beta/\beta_1$, $\beta_2 \equiv \beta_1$, and

$$\left|\frac{\beta_1}{\alpha\gamma\beta/\beta_2}\right| = |\eta_1^S, \eta_2^S| < |\Delta^5|.$$

Hence $\eta_1 \vdash_\mu \eta_2$, where

$$\begin{aligned}\mu &= R\left(q, \bar{q}', \frac{[\alpha\gamma\beta/\beta_1]}{[\beta_1]}, \frac{\beta_1^2}{(\alpha\gamma\beta/\beta_2)^2}, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\gamma\beta/\beta_2\}\right) \\ &= R\left(q, \bar{q}', \frac{[\langle\alpha\gamma\rangle]}{[\beta_1/(\alpha\gamma\beta/\beta_2)\langle\alpha\gamma\rangle]}, \frac{(\beta_1/(\alpha\gamma\beta/\beta_2)\langle\alpha\gamma\rangle)^2}{\langle\alpha\gamma\rangle^2}, \{\bar{\alpha}^2/\bar{\alpha} \equiv \langle\alpha\gamma\rangle\}\right)\end{aligned}$$

is in $\bar{M}$ by (6a). Suppose

$$\eta_1 = \left(\bar{q}, w, \frac{\beta_1^2}{\alpha\gamma\beta} \uparrow \left(\frac{\alpha\gamma\beta}{\beta_1}\right)^2\right).$$

Then $\alpha\gamma\beta/\beta_2 \leqslant \alpha\gamma\beta/\beta_1 \leqslant \beta_1 \leqslant \beta_2$, $\alpha\gamma\beta/\beta_1 \equiv \alpha\gamma\beta/\beta_2$, $\beta_1 \equiv \beta_2$, and

$$\left| \frac{\alpha\gamma\beta/\beta_1}{\alpha\gamma\beta/\beta_2} \right| = | \eta_1{}^S, \eta_2{}^S | < | \Delta^5 |.$$

Hence $\eta_1 \vdash_\mu \eta_2$ where

$$\mu = R\left(\bar{q}, \bar{q}', \frac{[\beta_1]}{[\alpha\gamma\beta/\beta_1]}, \left(\frac{\alpha\gamma\beta/\beta_1}{\alpha\gamma\beta/\beta_2}\right)^2, \{\bar{\alpha}^2/\bar{\alpha} \equiv \alpha\gamma\beta/\beta_2\}\right)$$

is in $\bar{M}$ by (6b). If $\eta_1 = (q', w, \alpha\gamma\beta/\beta_1{}^2 \uparrow \beta_1{}^2)$, then $\eta_1 \vdash_{\bar{M}} \eta_2$ by (6c). In any case, $\xi_2 \overset{m}{\vdash}_{\bar{M}} \eta_1 \vdash_{\bar{M}} \eta_2$, where $m \leqslant 2$.

(β) $| \eta_1{}^S, \eta_2{}^S | \geqslant | \Delta^5 |$. By techniques similar to those in (α), it is seen that $(\bar{p}', w, \alpha\gamma\beta/\alpha_3{}^2 \uparrow \alpha_3{}^2)$ serves as ξ_3, where

$$\alpha_3 = \frac{\alpha\gamma\beta}{\beta_2} \frac{\Delta^2}{\langle\gamma\rangle}.$$

LEMMA 4.3. *If* $(q_0, w, \uparrow) \overset{*}{\vdash}_M \eta$, *then there is a simulator* η_1 *of* η *such that* $(q_0, w, \uparrow) \vdash_{\bar{\mu}_1} \cdots \vdash_{\bar{\mu}_n} \eta_1$, *where each* $\bar{\mu}_i$ *is in* $\bar{M}$ *and at most eight consecutive* $\bar{\mu}_i$ *are generalized* ϵ*-moves. In particular, if* w *is in* $L(S)$, *then* w *is accepted by* $\bar{S}$ *with delay eight.*

Proof. We prove the following stronger proposition:

(i) If $(q_0, w, \uparrow) \overset{*}{\vdash}_M \eta$, then there is a simulator η_1 of η such that for each by-product η_2 of η_1, $(q_0, w, \uparrow) \vdash_{\bar{\mu}_1} \cdots \vdash_{\bar{\mu}_n} \eta_2$, where each $\bar{\mu}_i$ is in $\bar{M}$ and at most eight consecutive $\bar{\mu}_i$ are generalized ϵ-moves.

To show (i), it suffices to prove (ii) and (iii) below. (For then (i) follows by induction on $| w |$.)

(ii) Suppose that $\xi \vdash_M \eta$ by an input move in M and that ξ_1 is a simulator of ξ. Then there is a simulator η_1 of η such that, for each by-product η_2 of η_1, a by-product ξ_2 of ξ_1 can be found satisfying $\xi_2 \vdash_{\bar{M}} \eta_2$ by an input move of $\bar{M}$.

(iii) Suppose that $\xi \overset{*}{\vdash}_M \eta$ by ϵ-moves of M and that ξ_1 is a simulator of ξ. Then there is a simulator η_1 of η such that, for each by-product η_2 of η_1, a by-product ξ_2 of ξ_1 can be found satisfying $\xi_2 \overset{*}{\vdash}_{\bar{M}} \eta_2$ by at most eight generalized ϵ-moves of $\bar{M}$.

Item (ii) follows immediately from (1).

To prove (iii) let σ be the minimal length of the stack during the computation from

ξ to η. We shall assume that the right stack word becomes ϵ at least once during the computation. (If not, the argument is simpler.) Then

$$\xi = (p, w, \alpha \uparrow \beta) \overset{*}{\underset{M}{\vdash}} \theta = (r, w, \alpha\beta\uparrow) \text{ over } \alpha\beta,$$

$$\theta \overset{*}{\underset{M}{\vdash}} \zeta = (s, w, \sigma\uparrow) \text{ over } \sigma,$$

$$\zeta \overset{*}{\underset{M}{\vdash}} \psi = (t, w, \gamma\delta\uparrow) \text{ over } \sigma, \text{ and}$$

$$\psi \overset{*}{\underset{M}{\vdash}} \eta = (q, w, \gamma \uparrow \delta) \text{ over } \gamma\delta.$$

By Lemma 4.2, there is a simulator θ_1 of θ such that for, each by-product θ_2 of θ_1, a by-product ξ_2 of ξ_1 can be found satisfying $\xi_2 \overset{*}{\underset{\bar{M}}{\longmapsto}} \theta_2$ in at most three generalized moves. By Proposition2 4. and (3), ζ is a simulator ζ_1 of ζ such that, for each by-product ζ_2 of ζ_1, a by-product θ_2 of θ_1 can be found satisfying $\theta_2 \underset{\bar{M}}{\longmapsto} \zeta_2$. Similarly, by (2), ψ serves as a simulator ψ_1 of ψ such that, for each by-product ψ_2 of ψ_1, a by-product ζ_2 of ζ_1 can be found satisfying $\zeta_2 \underset{\bar{M}}{\longmapsto} \psi_2$. By Lemma 4.2 again, there is a simulator η_1 of η such that, for each by-product η_2 of η_1, a by-product ψ_2 of ψ_1 can be found satisfying $\psi_2 \overset{*}{\underset{\bar{M}}{\longmapsto}} \eta_2$ in at most three generalized moves. Clearly η_1 satisfies the requirements of (iii) and the proof of Lemma 4.3 is complete.

Finally, the third of the three lemmas:

LEMMA 4.4. *If* $(q_0, w_0, \uparrow) \overset{*}{\underset{\bar{M}}{\longmapsto}} \eta_1$, *then*

(α) *the state of* η_1 *is in* $\{q, \bar{q}/q \text{ in } K_0\}$ *and* $(q_0, w_0, \uparrow) \overset{*}{\underset{M}{\longmapsto}} \eta_1{}^S$, *or*

(β) *the state of* η_1 *is in* $\{q'/q \text{ in } K_0\}$ *and* $(q_0, w_0, \uparrow) \overset{*}{\underset{M}{\longmapsto}} \eta$ *for all* η *satisfying* $n \equiv \eta_1{}^S$ *and* $\eta_1{}^S \leqslant \eta$, *or*

(γ) *the state of* η_1 *is in* $\{\bar{q}'/q \text{ in } K_0\}$ *and* $(q_0, w_0, \uparrow) \overset{*}{\underset{M}{\longmapsto}} \eta$ *for some* η *satisfying* $\eta \equiv \eta_1{}^S$ *and* $\eta_1{}^S \leqslant \eta$.

In particular, $L(\bar{S}) \subseteq L(S)$.

Proof. Let us call an $\bar{S}$-configuration η_1 *valid* if $(q_0, w_0, \uparrow) \overset{*}{\underset{\bar{M}}{\longmapsto}} \eta_1$ and (α), (β), or (γ) holds. We shall prove the following statement:

(*) If η_1 is valid and $\eta_1 \underset{\bar{\mu}}{\longmapsto} \eta_2$, with $\bar{\mu}$ in $\bar{M}$, then η_2 is valid.

(Using induction and (*), the lemma obviously results.)

The proof of (*) is by examination of each of the generalized moves in $\bar{M}$. We give the argument for one of the complicated generalized moves, leaving the rest to the reader.

Suppose that η_1 is valid and $\eta_1 \vdash_{\bar{\mu}} \eta_2$, with $\bar{\mu}$ of the form (4h). Then

$$\eta_1 = \left(\bar{p}', w, \frac{\langle\gamma\rangle\beta}{\alpha} \uparrow \frac{\Delta^4}{\langle\gamma\rangle^2}\bar{\alpha}^2\right)$$

and

$$\eta_2 = \left(\bar{q}', w, \frac{\langle\gamma\rangle\beta}{\alpha} \frac{\Delta^4}{\langle\gamma\rangle^2} \uparrow \bar{\alpha}^2\right),$$

where $(p, w, \alpha\uparrow\gamma\beta) \vdash_M^* (q, w, \alpha\gamma\uparrow\beta)$ over $\alpha\gamma\beta$, $\bar{\alpha} \equiv \alpha\gamma$, $\kappa \leqslant \alpha$, and $\kappa \leqslant \beta$. We shall show that $(q_0, w_0, \uparrow) \vdash_M^* \eta$ for some η satisfying $\eta \equiv \eta_2$ and $\eta_2{}^S \leqslant \eta$.

Now

$$\eta_1{}^S = \left(p, w, \frac{\Delta^2}{\langle\gamma\rangle}\bar{\alpha} \uparrow \frac{\langle\gamma\rangle\beta}{\alpha} \frac{\Delta^2}{\langle\gamma\rangle}\bar{\alpha}\right) \equiv (p, w, \alpha \uparrow \gamma\beta)$$

and

$$\eta_2{}^S = \left(q, w, \bar{\alpha} \uparrow \frac{\langle\gamma\rangle\beta}{\alpha} \frac{\Delta^4}{\langle\gamma\rangle^2}\bar{\alpha}\right) \equiv (q, w, \alpha\gamma \uparrow \beta).$$

Since η_1 is valid, $(q_0, w_0, \uparrow) \vdash_M^* \xi$ for some ξ equivalent to $\eta_1{}^S$, $\eta_1{}^S \leqslant \xi$. Then $\xi = (p, w, \alpha_1 \uparrow \beta_1)$, with $\alpha_1\beta_1 = (\Delta^2/\langle\gamma\rangle)\,\bar{\alpha}(\langle\gamma\rangle\beta/\alpha)(\Delta^2/\langle\alpha\rangle)\bar{\alpha}$, $\alpha_1 \equiv (\Delta^2/\langle\gamma\rangle)\bar{\alpha} \equiv \alpha$, and $\beta_1 \equiv (\langle\gamma\rangle\beta/\alpha)(\Delta^2/\langle\gamma\rangle)\bar{\alpha} \equiv \gamma\beta \equiv \langle\gamma\rangle\langle\beta\rangle$. Since $\eta_1{}^S \leqslant \xi$, $(\Delta^2/\langle\gamma\rangle)\bar{\alpha} \leqslant \min(\alpha_1, \beta_1) \leqslant \alpha_1$ and $(\Delta^2/\langle\gamma\rangle)\bar{\alpha} \leqslant \beta_1$. Also, $\langle\gamma\rangle\langle\beta\rangle \leqslant (\kappa\pi)(\kappa\pi) \leqslant \Delta \leqslant \Delta(\Delta/\langle\gamma\rangle)\bar{\alpha}$ since $\epsilon \leqslant (\Delta/\langle\gamma\rangle)$. Thus $\langle\gamma\rangle\langle\beta\rangle \leqslant \beta_1$, so that $\beta \equiv \beta_1/\langle\gamma\rangle$. Since $\bar{\alpha}(\Delta^2/\langle\gamma\rangle) \leqslant \alpha_1$, $\bar{\alpha} \leqslant \bar{\alpha}\Delta^2 \leqslant \alpha_1\langle\gamma\rangle$. Since $(\Delta^2/\langle\gamma\rangle)\bar{\alpha} \leqslant \beta_1$ and $\epsilon \leqslant \Delta/\langle\gamma\rangle$, $\bar{\alpha} \leqslant (\Delta^2/\langle\gamma\rangle\langle\gamma\rangle)\bar{\alpha} \leqslant \beta_1/\langle\gamma\rangle$. Let $\eta = (q, w, \alpha_1\langle\gamma\rangle \uparrow \beta_1/\langle\gamma\rangle)$. Then $\eta \equiv \eta_2{}^S$ and, since $\bar{\alpha} \leqslant \alpha_1\langle\gamma\rangle$ and $\bar{\alpha} \leqslant \beta_1/\langle\gamma\rangle$, $\eta_2{}^S \leqslant \eta$. By Proposition 2.4, $\xi \vdash_M^* \eta$. Then $(q_0, w_0, \uparrow) \vdash_M^* \eta$ for some $\eta \equiv \eta_2{}^S$, with $\eta_2{}^S \leqslant \eta$.

This completes the argument for Lemma 4.4 and thus of Proposition 4.1.

5. The Main Result

In this section we finally establish our main result. First, though, we prove the third of the three main propositions.

Proposition 5.1. $\mathscr{L}_{\mathrm{SEQR}} \subseteq \mathscr{L}_{\mathrm{BQRG}}$.

Proof. Let $S = (M, q_0, q_1)$ be an *SE*-quasi-realtime stack-counter acceptor, with delay d. Let $\equiv_S$ (abbreviated $\equiv$) be a finite-index congruence relation on c^* with the properties guaranteed by Proposition 2.4. For each word α in c^* let $\langle\alpha\rangle$ be the shortest word in $[\alpha]$. Let π be the period and κ the constant of $\equiv$. Let $\Delta = \pi^{2|\pi\kappa|(d+1)}$.

We now define a generalized stack-counter acceptor $\bar{S} = (\bar{M}, q_0, q_1)$. For each state p in S let p' and p'' be new symbols. The roles of the states are to be as follows. The configuration $(p, w, \alpha\uparrow\beta)$ of $\bar{S}$ always simulates $(p, w, \alpha\uparrow\beta)$ of S. The configuration $(p', w, \alpha\uparrow\beta)$ of $\bar{S}$ simulates $(p, w, \alpha\uparrow\beta)$ of S when *all* configurations $(p, w, \alpha'\uparrow\beta)$, with $\alpha' \equiv \alpha$ and $\alpha' \geqslant \alpha$, are obtainable in S from the start state of S. The configuration $(p'', w, \alpha'\uparrow\beta)$ of $\bar{S}$ simulates $(p, w, \alpha\uparrow\beta)$ of S when *some* configuration $(p, w, \alpha\uparrow\beta)$, with $\alpha' \equiv \alpha$, and $\alpha' \geqslant \alpha$, is obtainable in S from the start state of S.

The set $\bar{M}$ is to consist of the generalized moves in (1)–(6) below (for all states p and q in S and all words α, β, γ in c^*, with $\alpha < \Delta^4$, $\beta < \Delta^4$, and $\gamma < \Delta^4$):

(1) If $I(p, q, a)$ is in M, then $\bar{M}$ contains $I(p, q, a)$, $I(p', q', a)$, and $I(p'', q'', a)$.

(2) If $(p, w, \alpha\uparrow) \vdash_M^* (q, w, \alpha\gamma\uparrow)$ over α, then $\bar{M}$ contains[13]

(a) $S(p, q, [\alpha], \gamma)$ and $S(p', q', [\alpha], \gamma)$,

(b) $S(p, q', [\alpha], \gamma)$, if $\kappa \leqslant \gamma$, and

(c) $E(p'', q'', [\alpha\gamma], \Delta^2/\langle\gamma\rangle)$, if $\kappa \leqslant \alpha$.

(3) If $(p, w, \alpha\gamma\uparrow) \vdash_M^* (q, w, \alpha\uparrow)$ over α, then $\bar{M}$ contains

(a) $E(p, q, [\alpha], \gamma)$ and $E(p'', q'', [\alpha], \gamma)$,

(b) $E(p'', q, [\alpha], \gamma)$, if $\kappa \leqslant \alpha$, and

(c) $S(p', q', [\alpha\gamma], \Delta^2/\langle\gamma\rangle)$, if $\kappa \leqslant \alpha$.

(4) If $(p, w, \alpha\uparrow\gamma\beta) \vdash_M^* (q, w, \alpha\gamma\uparrow\beta)$ over $\alpha\gamma\beta$, then $\bar{M}$ contains $R(p, q, [\alpha], \gamma, [\beta])$, $R(p', q', [\alpha], \gamma, [\beta])$, and $R(p'', q'', [\alpha], \gamma, [\beta])$.

(5) If $(p, w, \alpha\gamma\uparrow\beta) \vdash_M^* (q, w, \alpha\uparrow\gamma\beta)$ over $\alpha\gamma\beta$, then $\bar{M}$ contains $L(p, q, [\alpha], \gamma, [\beta])$, $L(p', q', [\alpha], \gamma, [\beta])$, and $L(p'', q'', [\alpha], \gamma, [\beta])$.

(6) If $\alpha\gamma \equiv \alpha$, then $\bar{M}$ contains

(a) $E(q, q'', [\alpha], \gamma)$, $E(q', q'', [\alpha], \gamma)$, and $E(q'', q'', [\alpha], \gamma)$,

(b) $S(p', p, [\alpha], \gamma)$, $S(p', p', [\alpha], \gamma)$, and $S(p', p'', [\alpha], \gamma)$,

(c) $\not{c}S(p', p, \gamma, \alpha, [\beta])$, $\not{c}S(p', p', \gamma, \alpha, [\beta])$, and $\not{c}S(p', p'', \gamma, \alpha, [\beta])$, and

(d) $\not{c}E(q, q'', \gamma, \alpha, [\beta])$, $\not{c}E(q', q'', \gamma, \alpha, [\beta])$, and $\not{c}E(q'', q'', \gamma, \alpha, [\beta])$.

Clearly $\bar{S}$ is a bounded generalized stack-counter acceptor. We shall see that $\bar{S}$ is quasi-realtime and $L(\bar{S}) = L(S)$. To do this we shall need some terminology for discussing the simulation of S by $\bar{S}$.

The terms *S-configuration*, *$\bar{S}$-configuration*, and *equivalent configurations* are the same as in Section 4. The partial ordering of configurations is different. For configurations ξ and η with the same right stack word, write $\xi \leqslant \eta$ if the stack word of ξ is $\leqslant$ the stack word of η.

[13] If A is a unit set, say $A = \{\nu\}$, then we write ν instead of $\{\nu\}$.

For each $\bar{S}$ configuration ξ of the form $(p, w, \alpha\uparrow\beta)$, $(p', w, \alpha\uparrow\beta)$, or $(p'', w, \alpha\uparrow\beta)$, call $(p, w, \alpha\uparrow\beta)$, denoted by ξ^S, the *corresponding S-configuration.*

By a *simulator* of an S-configuration $\xi = (p, w, \alpha\uparrow\beta)$ is meant any $\bar{S}$-configuration ξ_1 either equal to ξ or of the form $(p', w, \alpha_1\uparrow\beta)$, with $\alpha_1 \equiv \alpha$ and $\kappa \leqslant \alpha_1\beta \leqslant \alpha\beta$.

Let ξ_1 be any $\bar{S}$-configuration of the form $(p, w, \alpha\uparrow\beta)$ or $(p', w, \alpha\uparrow\beta)$. By a *by-product* of ξ_1 is meant any $\bar{S}$-configuration ξ_2 either $=\xi_1$ or of the form $(p'', w, \alpha_2\uparrow\beta)$, with $\alpha_2 \equiv \alpha$ and $\kappa \leqslant \alpha_2\beta \leqslant \alpha\beta$.

Using the preceding terminology, we present four lemmas. The arguments are similar to those in Section 4 but considerably simpler.

LEMMA 5.1. *Let $\xi \vdash_M^* \eta$ over α, where $\xi = (p, w, \alpha\uparrow)$ and $\eta = (q, w, \alpha\gamma\uparrow)$ ($\xi = (p, w, \alpha\gamma\uparrow)$ and $\eta = (q, w, \alpha\uparrow)$). Then, for every simulator ξ_1 of ξ, there is a simulator η_1 of η with the following property: For each by-product η_2 of η_1, there is a by-product ξ_2 of ξ_1 for which $\xi_2 \vdash_M^d \eta_2$ with $d \leqslant 3$.*

Proof. We shall prove only the case in which $\xi = (p, w, \alpha\uparrow)$ and $\eta = (q, w, \alpha\gamma\uparrow)$, an analogous argument holding for the other case.

Let ξ_1 be a simulator of ξ. Three situations arise:

(α) $\xi_1 = \xi$ and $\gamma < \kappa$. Then η is a suitable η_1. For consider any by-product η_2 of η_1. If $\eta_2 = \eta_1$, then $\xi_1 \vdash_M \eta_2$ by (2a). Suppose that $\eta_2 \neq \eta_1$. Then $\eta_2 = (q'', w, \beta\uparrow)$, where $\beta \equiv \alpha\gamma$ and $\kappa \leqslant \beta \leqslant \alpha\gamma$. If $\alpha\gamma/\beta < \Delta^2$, then $\xi_1 \vdash_M \eta_1 \vdash_M \eta_2$ by (2a) and (6a). If $\alpha\gamma/\beta \geqslant \Delta^2$, then $\xi_2 = (p'', w, \beta\Delta^2/\gamma\uparrow)$ is a by-product of ξ_1, and $\xi_2 \vdash_M \eta_2$ by (2c).

(β) $\xi_1 = \xi$ and $\gamma \geqslant \kappa$. Then, as is easily seen, $(q', w, \alpha\langle\gamma\rangle\uparrow)$ is a suitable η_1.

(γ) $\xi_1 \neq \xi$. Then $\xi = (p', w, \alpha_1\uparrow)$, where $\alpha_1 \equiv \alpha$ and $\kappa \leqslant \alpha_1 \leqslant \alpha$. As is easily seen, $(q', w, \alpha_1\langle\gamma\rangle\uparrow)$ is a suitable η_1.

LEMMA 5.2. *Let $\xi \vdash_M^* \eta$ over $\alpha\gamma\beta$, where $\xi = (p, w, \alpha\uparrow\gamma\beta)$ and $\eta = (q, w, \alpha\gamma\uparrow\beta)$ ($\xi = (p, w, \alpha\gamma\uparrow\beta)$ and $\eta = (q, w, \alpha\uparrow\gamma\beta)$). Then, for every simulator ξ_1 of ξ, there is a simulator η_1 of η with the following property: For each by-product η_2 of η_1, there is a by-product ξ_2 of ξ_1 for which $\xi_2 \vdash_M^d \eta_2$ with $d \leqslant 2$.*

Proof. We shall give only the proof for the case in which $\xi = (p, w, \alpha\uparrow\gamma\beta)$ and $\eta = (q, w, \alpha\gamma\uparrow\beta)$, the proof of the other case being analogous. Note that $\gamma \leqslant d$.

Let ξ_1 be a simulator of ξ. Two situations arise:

(α) $\xi_1 = \xi$. Then η is a suitable η_1. For consider any by-product η_2 of η_1. If $\eta_2 = \eta_1$, then $\xi_1 \vdash_M \eta_2$ by (4). Suppose that $\eta_2 \neq \eta_1$. Then $\eta_2 = (q'', w, \delta\uparrow\beta)$, where $\delta \equiv \alpha\delta$ and $\kappa \leqslant \delta\beta \leqslant \alpha\gamma\beta$. If $\alpha\delta \leqslant \Delta\delta$, then $\xi_1 \vdash_M \eta_1 \vdash_M \eta_2$ by (4) and (6d). If $\alpha\gamma > \Delta\delta$, then $\xi_2 = (p'', w, \Delta\delta/\gamma\uparrow\gamma\beta)$ is a by-product of ξ_1, and

$$\xi_2 \vdash_M (q'', w, \Delta\delta\uparrow\beta) \vdash_M \eta_2$$

by (4) and (6d). [Note that $| \Delta | > | \kappa | + d$. Hence $\Delta/\gamma > \kappa$, so that $\alpha > \Delta\delta/\gamma > \kappa$. Since $\delta \equiv \alpha\gamma \geqslant \kappa$, $\Delta\delta \equiv \delta$. Thus $\Delta\delta/\gamma = \alpha\Delta\delta/\alpha\gamma \equiv \alpha\Delta \equiv \alpha$.]

(β) $\xi_1 \neq \xi$. Then $\xi_1 = (p', w, \alpha_1\uparrow\gamma\beta)$, where $\alpha_1 \equiv \alpha$ and $\kappa \leqslant \alpha_1\gamma\beta \leqslant \alpha\gamma\beta$. By an argument similar to (α), it is seen that $(q', w, \alpha_1\gamma\uparrow\beta)$ is a suitable η_1.

LEMMA 5.3. *If* $(q_0, w, \uparrow) \vdash_M^* \eta$, *then there is a simulator* η_1 *of* η *such that* $(q_0, w, \uparrow) \vdash_{\bar{\mu}_1} \cdots \vdash_{\bar{\mu}_n} \eta_1$, *where each* $\bar{\mu}_i$ *is in* $\overline{M}$ *and at most ten consecutive* $\bar{\mu}_i$ *are generalized* ϵ*-moves. In particular, if* w *is in* $L(S)$, *then* we *is accepted by* $\bar{S}$ *with delay ten.*

The proof parallels that of Lemma 4.3 above and is omitted.

LEMMA 5.4. *If* $(q_0, w_0, \uparrow) \vdash_{\overline{M}}^* \eta_1$, *then* η_1 *has one of the following forms*:

(α) $(q, w, \alpha\uparrow\beta)$, *where* $(q_0, w_0, \uparrow) \vdash_M^* \eta_1$;

(β) $(q', w, \alpha\uparrow\beta)$, *where* $(q_0, w_0, \uparrow) \vdash_M^* \eta$ *for all* $\eta = (q, w, \bar{\alpha}\uparrow\beta)$ *satisfying* $\bar{\alpha} \equiv \alpha$ *and* $\alpha \leqslant \bar{\alpha}$; *or*

(γ) $(q'', w, \alpha\uparrow\beta)$, *where* $(q_0, w_0, \uparrow) \vdash_M^* \eta$ *for some* $\eta = (q, w, \bar{\alpha}\uparrow\beta)$ *satisfying* $\alpha \equiv \bar{\alpha}$ *and* $\alpha \leqslant \bar{\alpha}$.

In particular, $L(\bar{S}) \subseteq L(S)$.

Proof. Let "valid" and (*) be as in Lemma 4.4. The proof of (*) is by examination of each of the generalized moves in $\overline{M}$. We give the argument for one of the complicated generalized moves, leaving the rest to the reader.

Suppose that η_1 is valid and $\eta_1 \vdash_{\bar{\mu}} \eta_2$, with $\bar{\mu}$ of the form (2c). Then $\eta_1 = (p'', w, \beta\Delta^2/\langle\gamma\rangle\uparrow)$ and $\eta_2 = (q'', w, \beta\uparrow)$, where $(p, w, \alpha\uparrow) \vdash_M^* (q, w, \alpha\gamma\uparrow)$ over α, $\beta \equiv \alpha\gamma$, and $\kappa \leqslant \alpha$. We shall show that $(q_0, w_0, \uparrow) \vdash_M^* \eta$ for some $\eta = (q, w, \bar{\beta}\uparrow)$ satisfying $\bar{\beta} \equiv \beta$ and $\bar{\beta} \geqslant \beta$. Since η_1 is valid, $(q_0, w_0, \uparrow) \vdash_M^* \xi$ for some configuration $\xi = (p, w, \bar{\alpha}\uparrow)$ with $\bar{\alpha} \equiv \beta\Delta^2/\langle\gamma\rangle$ and $\bar{\alpha} = \beta\Delta^2/\langle\gamma\rangle$. Since $\bar{\alpha} \equiv \alpha\gamma(\Delta^2/\langle\gamma\rangle) \equiv \alpha\Delta^2 \equiv \alpha$, $(q_0, w_0, \uparrow) \vdash_M^* \xi \vdash_M^* (q, w, \bar{\alpha}\langle\gamma\rangle\uparrow)$ by Proposition 2.4. Moreover, $\bar{\alpha}\langle\gamma\rangle \equiv \alpha\gamma \equiv \beta$ and $\bar{\alpha}\langle\gamma\rangle \geqslant \beta\Delta^2 \geqslant \beta$. Thus $(q, w, \bar{\alpha}\langle\gamma\rangle\uparrow)$ serves as a suitable η.

From Lemmas 5.3 and 5.4, and the fact that $\bar{S}$ is bounded, Proposition 5.1 follows.

We are now ready for the main result.

THEOREM $\mathscr{L} = \mathscr{L}_{QR}$.

This follows from Propositions 3.1, 4.1, and 5.1, and the fact that $\mathscr{L}_{QR} \subseteq \mathscr{L}$.

Remark. A (quasi-realtime) stack-counter acceptor with no right shifts and no left shifts is called a *(quasi-realtime) counter acceptor* [6]. By a modification of the argument in the paper, it is readily seen that any language L accepted by a counter acceptor is accepted by some quasi-realtime counter acceptor (so that the counter acceptor languages and the quasi-realtime counter acceptor languages coincide). To see this note that $L = L(S)$ for some SE-quasi-realtime stack-counter acceptor S with no right or left shifts. By Proposition 5.1, $L = L(S_1)$ for some bounded quasi-realtime generalized stack-counter acceptor S_1. Furthermore, S_1 is restricted to generalized input, generalized store, and generalized erase moves. (This is because (1), (2), and (3) of Section 5 involve only generalized input, generalized store, and generalized erase moves. Items (4), (5), and (6), when used for a counter, introduce only generalized moves which can be realized by generalized store or generalized erase moves.) By Proposition 3.1, $L = L(S_2)$ for some quasi-realtime stack-counter acceptor S_2. Furthermore, S_2 is a quasi-realtime counter acceptor since right and left shifts arise only through (4)–(7) of Section 3 and thus do not occur in the present situation.

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